EFFECT OF THE FRACTIONAL DERIVATIVES ON UNSTEADY NATURAL CONVECTION OF A NANOFLUID IN AN ENCLOSURE

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***Abstract*— This paper studies effect of the time and space fractional derivatives orders on the unsteady fluid flow and heat transfer inside a square enclosure filled with Cu-H_2O nanofluid. An active part is located in the bottom wall of the enclosure and theory of the conformable fractional derivatives is applied on the time and space derivatives. The governing fractional partial differential equations are solved numerically using the finite difference method and the obtained results are presented in terms of the streamlines, isotherms, velocity component, local and average Nusselt numbers. The results revealed that the local and average Nusselt numbers are enhanced as either the time fractional derivatives order or the space fractional derivatives order decreases. Also, effects of variations of the time fractional derivatives order are significant only at the low values of the time parameter.**

***Keywords*— Natural convection; enclosure; nanofluid; conformable fractional derivative; finite difference.**

I. INTRODUCTION

Heat transfer by convective transport attracted the attention of a lot of researchers due to its important practical applications in science and engineering. These applications include for example cooling electronic system, building insulation, solar energy collection, components in the electrical and nuclear industries and cooling of heat-generating, (Incropera, 1988; Hoogendooren and Afgan, 1978; Cha and Jauria, 1984; Imberger and Hamblin, 1982). Kandaswamy *et al.* (2007) studied effect of the different values of the Grashof number, position of the heated plat and different aspect ratios on the natural convection in a square cavity. They found that the heat transfer rate is reduced in both the vertical and horizontal positions of the plate as Gr increases. Also, the heat transfer decreases when aspect ratio of the heated thin plate is decreased. Performance of the heat transfer and entropy generation of the natural convection in a nanofluid-filled U-shaped cavity was studied by Cho *et al.* (2015). The results show that when the Rayleigh number increases, the Nusselt number and the total entropy generation are increased. Effect of the size of the heater in a square enclosure on the heat transfer was studied by Ragui *et al.* (2013). They found that when the Rayleigh number increases, the heat transfer is supported. An *et al.* (2013) obtained a hybrid numerical solution for the natural convection in a cavity with volumetric heat generation. Mansour and Ahmed (2015) studied the natural convection heat transfer in an inclined triangular enclosure filled with Cu-water nanofluid and saturated by a porous medium. Mansour *et al.* (2014a) studied the natural convection fluid flow and heat transfer between two cavities filled with a water-based nanofluid using the finite difference method. In Mansour *et al.* (2014b), the free convection fluid flow and heat transfer inside C-shaped enclosures filled with a Cu-water nanofluid was discussed numerically using the finite difference method. Ahmed and Aly (2019) used the ISPH method to study the natural convection of a nanofluid in an enclosure filled with solid particles within an inner cross shape. The result showed that the decrease in the cross shape lengths by 0.6 increases values of the stream function by 27.8%. Also, the cold and moving solid particles give the higher rate of the heat transfer comparing with case of the fixed and cold solid particles. Raizah *et al.* (2018) presented the natural convection flow of non-Newtonian nanofluids in a slanted cavity. The cavity is open, shallow and filled with porous media. The results disclosed that when the power-index n is increased, rate of the heat transfer is reduced while the average Bejan number is enhanced. Aly *et al.* (2018) studied the mixed convection in a cavity saturated with a wavy layer porous medium. The study shows that the increase in the Darcy number brings a big resistance force for the fluid flow and hence increases the heat transfer. Also, the average Bejan number is close to unity at the forced convection mode.

The fractional calculus became an important branch in the pure and applied mathematics. The practical applications of this topic appears in control theory of dynamical systems, nanotechnology and viscoelasticity (Katugampola, 2014; Baleanu *et al.*, 2010; Monje *et al.*, 2010; Caponetto *et al.*, 2010; Mainardi, 2010). Many researchers give a definition of the fractional derivative. The most popular definitions are Riemann-Liouville, Caputo, Riesz and Grünwald-Letnikov (see Oldham and

Spanier, 1974; Miller, 1993; Kilbas *et al.*, 2006; Podlubny, 1999). The fractional governing equations for the fluids flow are obtained from those of the integer case by substituting the derivatives of an integer order with the fractional derivatives order α. For example, $\alpha = 1$ corresponds to the classical diffusion whereas for $0 < \alpha < 1$, the transport phenomena exhibits the sub diffusion and the case of $\alpha > 1$ exhibits the super diffusion. Many of the usual properties of the ordinary (integer) derivatives such as product, quotient and chain rules are not provided for the fractional derivatives. So, the researchers found some difficulties in using the algebraic operations in the non-integer calculus. For these reasons, it was appeared a new definition that well-behaved simply. It is called "the conformable fractional derivative" and it is depending just on the basic limit definition, (see Unal *et al.*, 2015; Abu Hammad and Khalil, 2014a; 2014b; Khalil *et al.*, 2014). These references used the limits to introduce the conformable fractional derivative in the form:

$$D^{\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\beta}\right) - f(t)}{\varepsilon} \quad \forall t > 0, \beta(0,1],$$

$$f^{(\beta)}(0) = \lim_{t \to 0^{+}} f^{(\beta)}(t).$$

Also, the conformable fractional derivative has the following properties:

$$D^{\beta} t^{p} = p t^{p-\beta}, p \in N, D^{\beta} c = 0, \forall f(t) = c.$$

$$D^{\beta}(af + bg) = aD^{\beta} f + bD^{\beta} g, \forall a, b \in Z$$

$$D^{\beta}(fg) = fD^{\beta} g + fD^{\beta} g,$$

$$D^{\beta} f(g) = \frac{df}{dg} D^{\beta} g, \;\; D^{\beta} f(g) = t^{1-\beta} \frac{df}{dg},$$

Abdeljawad (2015) improved the conformable fractional definition and presented the basic concepts of this new simple fractional calculus. Iyiola and Nwaeze (2016) introduced some results on the recently proposed conformable fractional derivatives and integral. They also applied the D'Alambert approach to the conformable fractional differential equation as an application. Ahmed *et al.* (2019) studied effect of the fractional parameters α and β on the natural convection in a slanted cavity filled with a porous medium using the conformable fractional derivative. The study interested with effects of α, β at different values of the Rayleigh number and the inclination angle. The result revealed that the increase in the inclination angle gives a clear reduction in both of the fluid flow and heat transfer. Also, the convection is better in case of higher values of the Rayleigh number.

The main aim of this paper is to study the natural convention inside an enclosure filled with nanofluid under effect of the conformable fractional derivative. The definition of the conformable fractional derivatives is used to treat the time and space fractional derivatives. The resulting equations are solved numerically using the finite difference method. The results show that effects of the parameter β on the streamlines, isothermal, local Nusselt and average Nusselt numbers are significant in case of the low values of the time parameter. Also, this paper provides a detailed discussion as well as a graphical representation of all obtained results.

Table 1. Thermo-physical properties of water and nanoparticles.

	water	Copper(Cu)
$\rho(kgm^{-3})$	997.1	8933
$C_p(Jkg^{-1}K^{-1})$	4179	385
$k(Wm^{-1}K^{-1})$	0.613	401
$\beta(K^{-1})$	21×10^{-5}	1.67×10^{-5}

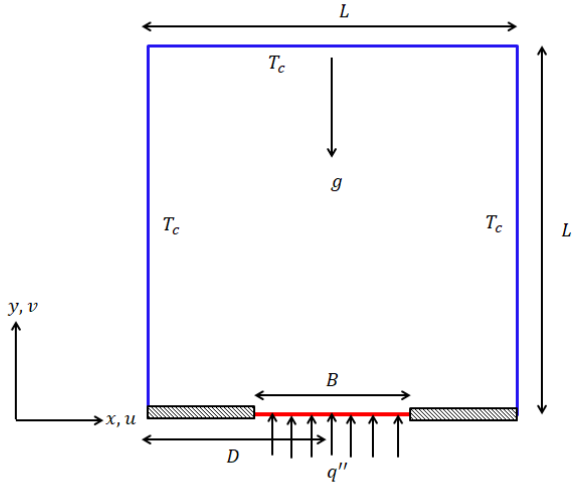

Fig. 1. Physical model of the problem.

II. METHODS

A. Problem description

Let us consider an unsteady two-dimensional natural convection flow inside a square cavity of length L that is filled with a nanofluid, as shown in Fig. 1. A heat source with length B is located on the lower wall. The worked nanofluid is assumed to be incompressible, laminar and the base fluid (water) and the solid spherical nanoparticles (Cu) are in the thermal equilibrium model. The thermo-physical properties (Table 1) of the nanofluid are assumed constants except the density variations, which are determined based on the Boussinesq approximation.

B. Mathematical formulation

The continuity, momentum and energy equations for the laminar and unsteady natural convection in the two-dimensional enclosure can be written in dimensional form as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_{nf}}\left(-\frac{\partial P}{\partial x} + \mu_{nf}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right) \tag{2}$$

$$\frac{\partial v}{\partial \tau} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho_{nf}}\left(-\frac{\partial P}{\partial y} + \mu_{nf}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g(\rho\beta)_{nf}(T - T_e)\right) \tag{3}$$

$$\frac{\partial T}{\partial \tau} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

The conformable fractional derivative of the previous system can be written as:

$$D_x^{\alpha} u + D_y^{\alpha} v = 0 \tag{5}$$

$$D_{\tau}^{\beta} u + uD_x^{\alpha} u + vD_y^{\alpha} u = \frac{1}{\rho_{nf}}\left(-D_x^{\alpha} p + \mu_{nf}\left(D_x^{\alpha}(D_x^{\alpha} u) + D_y^{\alpha}(D_y^{\alpha} u)\right)\right) \tag{6}$$

$$D_\tau^\beta v + uD_x^\alpha v + vD_y^\alpha v = \frac{1}{\rho_{nf}}\big(-D_y^\alpha p + \mu_{nf}\big(D_x^\alpha(D_x^\alpha v) + D_y^\alpha\big(D_y^\alpha v\big) + (\rho\beta)_{nf} g(T - T_c)\big)\big) \quad (7)$$

$$D_\tau^\beta T + uD_x^\alpha T + vD_y^\alpha T = \alpha_{nf}\left(D_x^\alpha(D_x^\alpha T) + D_y^\alpha\big(D_y^\alpha T\big)\right) \quad (8)$$

where, $D^\propto$ is conformable fractional derivative operator. The boundary conditions are:
for $y = 0$,

$$u = v = 0,$$

$$\frac{\partial T}{\partial y} = \frac{q''}{k_{nf}}, (D - 0.5B) \le \frac{x}{L} \le (D + 0.5B)$$

$$\frac{\partial T}{\partial y} = 0, \text{ otherwise.}$$

for $y = L$ and $0 \le x \le 1$

$$u = v = 0, T = T_c$$

for $x = L$ and $0 \le y \le 1$

$$u = v = 0, T = T_c$$

for $x = 0$ and $0 \le y \le 1$

$$u = v = 0, T = T_c \quad (9)$$

The thermophysical properties of the nanofluid are considered as functions of the nanoparticles volume fractions and those are given as

The effective density of the nanofluid is given as:

$$\rho_{nf} = (1 - \varphi)\rho_f + \varphi\rho_p \quad (10)$$

where ϕ is the solid volume fraction of the nanofluid, ρ_f and ρ_p are the densities of the fluid and nanoparticles respectively. Additionally, the heat capacitance of the nanofluid is given by:

$$\big(\rho c_p\big)_{nf} = (1 - \varphi)\big(\rho c_p\big)_f + \varphi\big(\rho c_p\big)_p \quad (11)$$

Further, the thermal expansion coefficient of the nanofluid can be determined by:

$$(\rho\beta)_{nf} = (1 - \varphi)(\rho\beta)_f + \varphi(\rho\beta)_p \quad (12)$$

where β_f and β_p are coefficients of the thermal expansion of the fluid and the nanoparticles, respectively. Moreover, the thermal diffusivity, α_{nf} of the nanofluid is:

$$\alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}} \quad (13)$$

where k_{nf} is the thermal conductivity of the nanofluid and this parameter is determined for the spherical nanoparticles according to the Maxwell-Garnetts (Maxwell, 1904) model as:

$$\frac{k_{nf}}{k_f} = \frac{(k_p+2k_f) - 2\phi(k_f - k_p)}{(k_p+2k_f) + \phi(k_f - k_p)} \quad (14)$$

The effective dynamic viscosity of the nanofluid is based on the Brinkman (1952) model and it is given by:

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \quad (15)$$

where μ_f is the viscosity of the base fluid.

Introducing the following dimensionless variables:

$$X = {}^{x}/_{L}, Y = {}^{y}/_{L}, U = {}^{uL}/_{\alpha_f}, V = {}^{vL}/_{\alpha_f},$$
$$\tau = {}^{\alpha_f t}/_{L^2}, P = \frac{pL^2}{\rho_{nf}\alpha_f^2}, \theta = \frac{T - T_c}{\Delta T}, \Delta T = \frac{q''L}{k_f} \quad (16)$$

Substituting Eq. (16) into Eqs. (5)- (8), the following dimensionless forms of the governing equations are obtained:

$$D_X^\alpha U + D_Y^\alpha V = 0 \quad (17)$$

$$D_\tau^\beta U + UD_X^\alpha U + VD_Y^\alpha U = -D_X^\alpha p + \frac{\mu_{nf}}{\rho_{nf}\alpha_f}\big(D_X^\alpha(D_X^\alpha U) + D_Y^\alpha(D_Y^\alpha U)\big) \quad (18)$$

$$D_\tau^\beta V + UD_X^\alpha V + VD_Y^\alpha V = -D_Y^\alpha p + \frac{\mu_{nf}}{\rho_{nf}\alpha_f}\big(D_X^\alpha(D_X^\alpha V) + D_Y^\alpha(D_Y^\alpha V)\big) + \frac{(\rho\beta)_{nf}}{\rho_{nf}\beta_f} Ra\Pr\theta \quad (19)$$

$$D_\tau^\beta \theta + UD_X^\alpha \theta + VD_Y^\alpha \theta = \frac{\alpha_{nf}}{\alpha_f}\big(D_X^\alpha(D_X^\alpha \theta) + D_Y^\alpha(D_Y^\alpha \theta)\big) \quad (20)$$

where

$$\Pr = \frac{\nu_f}{\alpha_f}, Ra = \frac{g\beta_f L^3\Delta T}{\nu_f\alpha_f}, \Delta T = \frac{q''L}{k_f}$$

are the Prandtl number, the Rayleigh number and the temperature difference, respectively. Further, the dimensionless boundary conditions for Eqs. (18-20) are expressed as:

$Y = 0, U = V = 0,$

$$\frac{\partial\theta}{\partial Y} = \frac{q''}{k_{nf}}, (D - 0.5B) \le X \le (D + 0.5B)$$

$$\frac{\partial\theta}{\partial Y} = 0, \text{ otherwise.}$$

for $Y = L$ and $0 \le X \le 1$

$$U = V = 0, \theta = 0$$

for $X = L$ and $0 \le Y \le 1$

$$U = V = 0, \theta = 0$$

for $X = 0$ and $0 \le Y \le 1$

$$U = V = 0, \theta = 0 \quad (21)$$

The local Nusselt number is defined as:

$$Nu_s = \frac{1}{(\theta)_{heat\,source}} \quad (22)$$

and the average Nusselt number is defined as:

$$Nu_m = \left(\frac{1}{B}\int_{D-0.5B}^{D+0.5B} Nu_s dX\right)_{Y=0} \quad (23)$$

Equations (17)-(20) with the boundary conditions (21) are solved numerically using the finite difference method (FDM). The theory of the conformable fractional derivatives is invoked in the Eqs. (17)-(20) then the forward difference approach is applied for the first order derivatives of the time and space while the second derivative is treated using the central difference approach. The resulting algebraic system is solved using the successive under relaxation (SUR) method and a relative error of order 10^{-6} is used as a convergence criteria.

C. Results and discussion

In this section, demonstration of the obtained results is presented. Here the governing parameters are considered in wide ranges, namely, the time fractional derivatives order β is vried from 0.95 to 0.8, the space fractional derivatives order α is varied from 1.0 to 0.75 and the referenced case is considered as $\varphi = 0.04, D = 0.5$, $Ra = 3 \times 10^5, B = 0.4, \beta = 0.95$.

Table 2 contains values the average Nusselt number for different values of the time and its fractional derivatives order β The table revealed that impacts of β is significant at the low values of τ_{max} and this effect is reduced as τ_{max} is grown. Also, for all values of τ_{max}, the reduction in β causes a diminution in values of Nu_m.

In Fig. 2, maps of the flow features (streamlines) and temperature distributions (isotherms) for the variations of

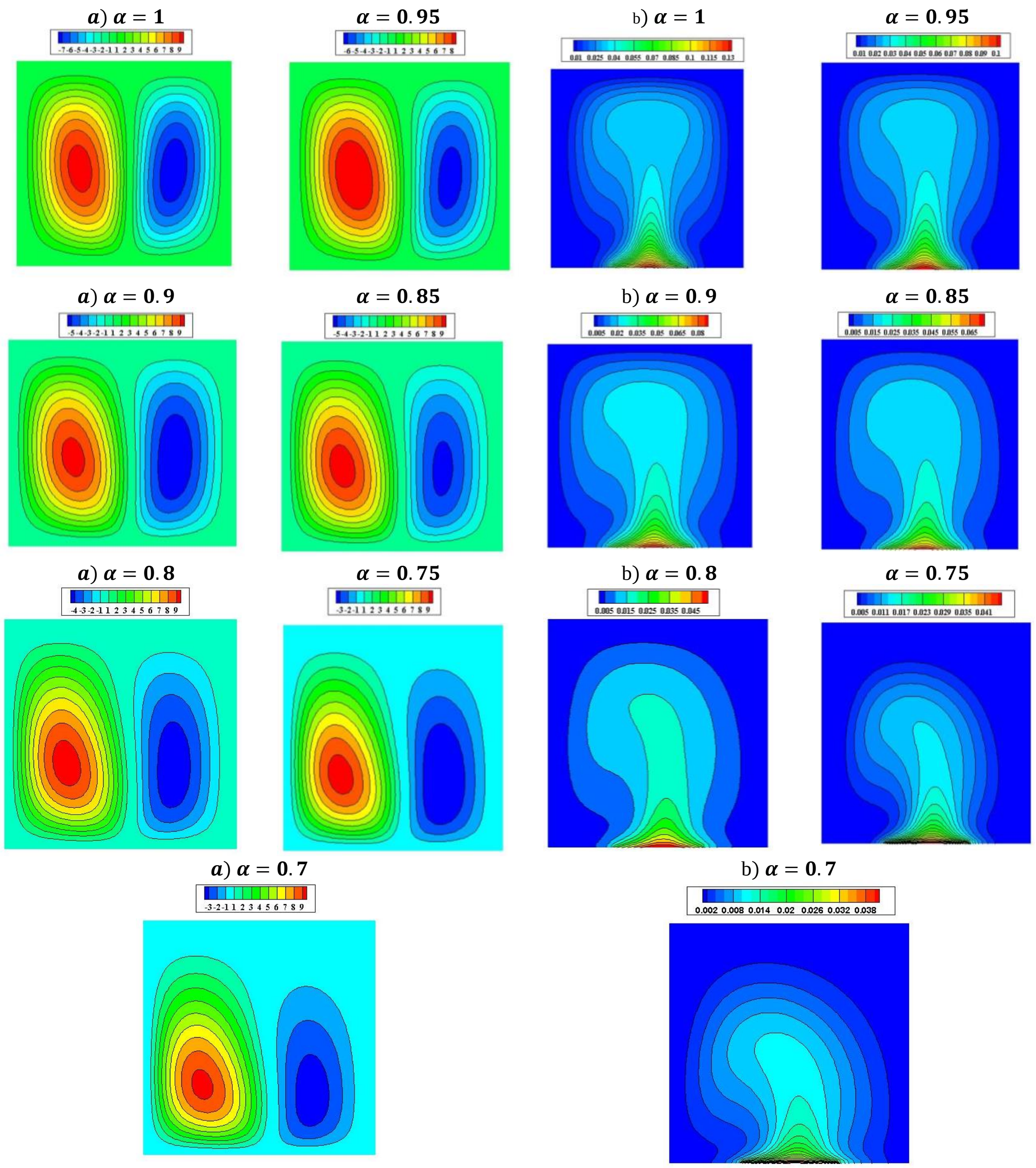

Fig. 2. a- Streamlines and b- Isothermal for Cu-water at $\tau = 0.15, \varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5, \beta = 0.95$.

the space fractional derivatives order α are depicted. Note, the fractional derivatives are taken on the derivatives with respect to X and Y. The results disclosed that activity of the flow is reduced as α is reduced. Also, the flow is concentrated in the left hand side of the enclosure as α is decreased. In the same context, there is a thermal zone near the heat source is noted for all values of α while as α is decreased, both of the temperature distributions and maximum values of the temperature are decreased.

Figures 3-6 show profiles of the local Nusselt number along the heat source, horizontal and vertical velocity components at the enclosure mid-section and the average Nusselt number for the different values of the time and space fractional derivatives orders α and β. The figures revealed that the local Nusselt number is enhanced as either α or β is reduced due to the inverse relation between the local Nusselt number and the maximum temperature (Eq. 22). In addition, there are maximum values for the vertical velocity component at the center line of the enclosure and these values are reduced as α decreases. Further, values of the average Nusselt is enhanced as the space fractional derivatives order α is diminished.

III. CONCLUSIONS

Unsteady natural convective flow and thermal fields in a square enclosure under impacts of the fractional derivatives orders on the time and space was performed in this paper. Definitions and theory of the conformable fractional derivatives are invoked in this study and the finite difference method is applied to solve the dimensionless governing equations. The important findings from this investigation are summarized as:

- The streamlines concentrated gradually in the left side of the cavity and the isotherms increases near the bottom wall.

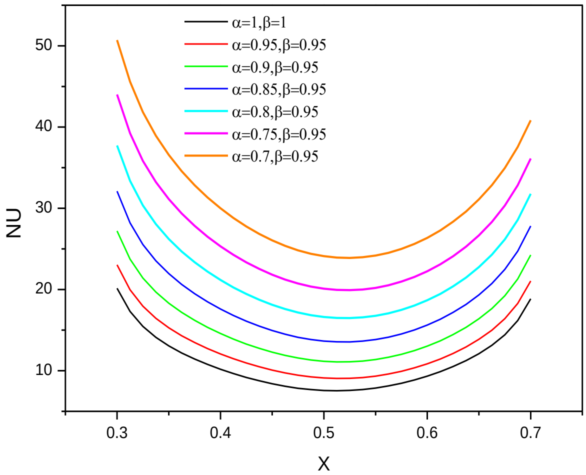

Fig. 3. Profiles of the local Nusselt number for Cu-water at $\tau = 0.15, \varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5$.

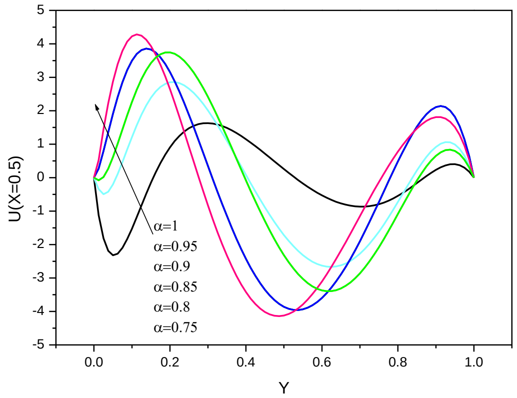

Fig. 4. Profiles of the local Nusselt number for Cu-water at $\varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5, \beta = 0.95$.

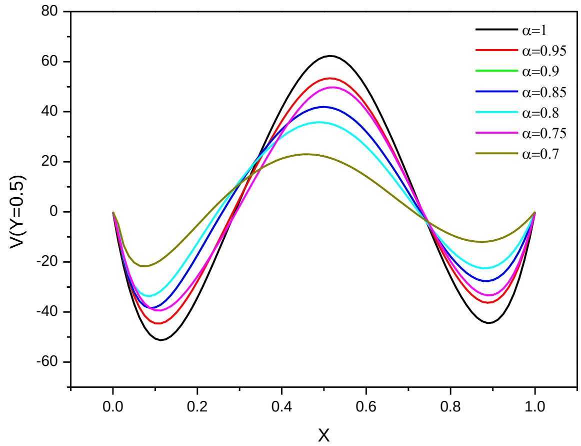

Fig. 5. Vertical velocity along the mid-section of the enclosure for Cu-water at $\varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5, \beta = 0.95$.

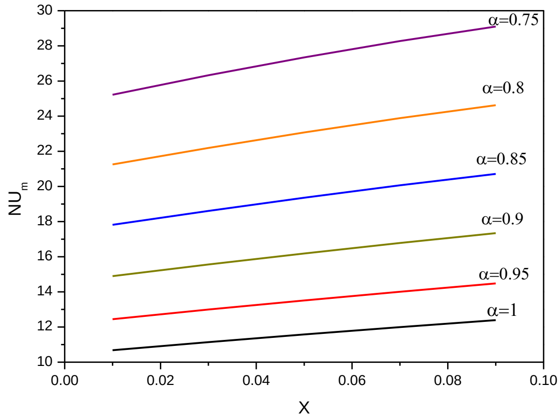

Fig. 6. Variation of the average Nusselt number for Cu-water at $\tau = 0.15, \varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5$.

Table 2. Values of the average Nusselt Nu_m for variations of τ_{max} and β at $\varphi = 0.04, D = 0.5, B = 0.4, Ra = 3 \times 10^5$, $\beta = 0.95$.

τ_{Max}	β	Num
0.01	0.95	19.4906
	0.9	18.4436
	0.85	17.74222
	0.80	17.28099
0.02	0.95	14.55306
	0.9	13.80804
	0.85	13.31111
	0.80	12.98539
0.05	0.95	10.30703
	0.9	9.87279
	0.85	9.590626
	0.80	9.409862
0.1	0.95	8.764265
	0.9	8.673367
	0.85	8.659124
	0.80	8.673273
0.2	0.95	9.615826
	0.9	9.623578
	0.85	9.59418
	0.80	9.565308
0.5	0.95	9.428584
	0.9	9.427379
	0.85	9.427791
	0.80	9.428111

- Effects of the time fractional order are clear in case of the low values of the time.
- Local and average Nusselt numbers are enhanced as either the time fractional derivatives order or the space fractional derivatives order decreases.
- The horizontal velocity increases with the increase in the fractional parameter α
- The horizontal velocity in the right hand side of the enclosure is greater than those of the left hand side and this explained the movement of the fluid in the enclosure from the right to the left.
- The variations in the vertical velocity were symmetric along the mid-section of the enclosure
- The average Nusselt number is increased when α is reduced, regardless values of the Rayleigh number.
- The fluid became more stable in left hand side.

ACKNOWLEDGEMENTS

The authors would like to express their gratitude to King Khalid University, Saudi Arabia for providing administrative and technical support

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Received July 2, 2018
Sent to Subject Editor May 5, 2019
Accepted October 7, 2019
Recommended by Subject Editor Fabio Giannetti