A NEW REFINED THEORY OF PLATES WITH TRANSVERSE SHEAR DEFORMATION FOR MODERATELY THICK AND THICK PLATES

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***Abstract*— In this paper we propose a new refined shear deformation plate theory which possesses a series of desirable features, the most salient of which are as follows: (1) loads, which are generally applied on the middle surface of the plate, act on the upper surface of the plate; (2) equations applicable to the calculation of the stresses in isotropic plates which provide the same order of accuracy as several theories with second-order shear deformation effects; (3) a theory, in the classical sense defined, since it gives easy expressions for application to problems in different fields in architecture and civil engineering.**

***Keywords*— Thick plates, first-order shear deformation theory, moderately thick plates.**

I. INTRODUCTION

A rectangular plate is usually considered thin if its thickness is less than a tenth of the minor dimension. When this limitation is not fulfilled it is a moderately thick (a term introduced by Love (1944)) or thick plate.

Since Reissner (1945), Mindlin (1951), Hencky (1947) and Reismann (1988) elaborated their technical calculation theories (first-order shear deformation theory), with the objective of widening the field of application of plates, many authors have studied this important field of structural mechanics.

Among other works, the current trend in the study of plates can be deduced from the themes of the articles collected by Voyiadjis and Karamanlidis (1990) and Kienzler *et al.* (2004). In the first publication it can be seen that four papers make a direct reference to moderately thick plates. The second one discusses common roots of different new plate and shell theories, reviewing the current state of the art and higher-order shear deformation plate theories including zigzag theories, global-local higher-order deformation theories and layer-wise laminated plate theories.

A more recent work by Ghugal and Shimpi (2002) is an in-depth study of the different theories on isotropic and anisotropic plates.

In this paper, we present a shear deformation plate theory which can be classified, if we follow Ghugal and Shimpi (2002), within the theories based on the semi-inverse method in the sense that the in-plane displacements are deduced through constitutive equations and assuming a distribution for the shear stresses.

Authors like Donnell (1976), Kromm (1953), Panc (1975) and Kienzler (2004) have followed this method for moderately thick plates, obtaining analytical solutions in many practical cases.

As noted in this work, not many authors have extended these studies to include thick plates.

The works of Voyiadjis *et al*. (1981, 1987, 1990) and Baluch and Voyiadjis (1984) show different applications of this method to thick plates but we found no analytical solutions based on Fourier series.

Since the 1980s, many higher-order plate theories have been developed. Perhaps, the most popular third-order plate theory is Reddy's (Reddy, 1990). We must also mention Touratier (1991) (transverse strain distribution as a sine function), Soldatos (1992) (hyperbolic shear deformation theory) and Shi (2007). Shi's theory was similar to Reddy´s in the sense that he used a parabolic distribution of transverse shear strain.

All these theories are characterized by the need for a high level of mathematical complexity to obtain solutions. Furthermore, the problems which are solved analytically only constitute several specific examples.

In this paper, we deduced new equations for moderately thick and thick plates. Like Reddy (1990) and Shi (2007), we also propose a parabolic distribution of shear strain through the thickness. Unlike them, however, and taking into account the semi-inverse method, we start not with the displacement field but with the distribution of shear strains through the thickness. Other interesting features of the theory are presented below

II. HYPOTHESIS AND OBTAINING GOVERNING EQUATIONS

The hypotheses to consider are as follows:

1) The loads, which are distributed, act at the upper surface of the plate in the downward z-direction and are perpendicular to the middle surface, and the displacements of the points located in the middle surface are also sensibly perpendicular to the middle surface (which is practically inelastic), that is, $\hat{u} \approx 0$ and $\hat{v} \approx 0$ where $\hat{u}$ and $\hat{v}$ are the displacements according to the x- and y-axes of the points located in the middle surface. The deflection w according to the z-axis of a generic point not located in the middle surface is given by

$$w = \hat{w} + f(x)\,, \tag{1}$$

where $\hat{w}$ is the deflection according to the z-axis of the points located in the middle surface. In the longitudinal deformation calculation subject to the thickness, Poisson's effect is not considered; that is,

$$\varepsilon_x \approx \frac{\sigma_x}{E} \approx 0 \tag{2}$$

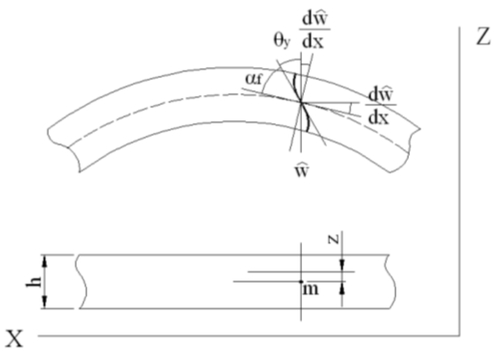

Figure 1. Assumed deformation patterns of the transverse normals in higher-order displacement-based theories.

2) The *fibres pertaining to the plate, which are straight and perpendicular to the middle surface before the deformation*, do not continue to be perpendicular, and they bend in such a way that the shearing strains are defined by a parabolic distribution throughout the thickness, given by

$$\gamma_{xz} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\gamma}_{xz};\ \gamma_{yz} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\gamma}_{yz} \qquad (3)$$

where $\hat{\gamma}_{xz}$ and $\hat{\gamma}_{yz}$ are the shearing strains in the points of the middle surface.

3) The rotation w_{xy} of a differential element around the z-axis is null for all points of the plate,

$$w_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) = 0. \qquad (4)$$

This important condition is deduced analytically if we establish the equilibrium of a plate element in its deformed configuration and take into consideration Reissner´s kinematic assumptions (Martínez Valle, 2012).

We denote by $\hat{w}$ the deflection, according to the z-axis, of *the generic point m located on the plate's mid-surface*, ϑ_x the angle rotated by the rectilinear segment normal to middle surface around the ox-axis, and ϑ_y the angle rotated around the oy-axis. According to its definition, the *shearing strain* in the xz-surface *at the point m located on the middle surface* ($\hat{\gamma}_{xz}$) is

$$\hat{\gamma}_{xz} = \frac{\pi}{2} - \alpha_f, \qquad (5)$$

which, as we can see in Fig. 1, is

$$\hat{\gamma}_{xz} = \vartheta_y + \frac{\partial \hat{w}}{\partial x}. \qquad (6)$$

Likewise, we deduce

$$\hat{\gamma}_{yz} = -\vartheta_x + \frac{\partial \hat{w}}{\partial y}, \qquad (7)$$

and we may also write

$$\hat{\gamma}_{xz} = \frac{\hat{\tau}_{xz}}{G} \text{ and } \hat{\gamma}_{yz} = \frac{\hat{\tau}_{yz}}{G}\ , \qquad (8)$$

where $\hat{\tau}_{xz}$ and $\hat{\tau}_{yz}$ are the shearing stresses at *the points located on the middle surface*.

Therefore shearing stresses at the points of the middle surface are

$$\frac{\hat{\tau}_{xz}}{G} = \vartheta_y + \frac{\partial \hat{w}}{\partial x}\ ;\ \frac{\hat{\tau}_{yz}}{G} = -\vartheta_x + \frac{\partial \hat{w}}{\partial y}, \qquad (9)$$

The shearing strains at a generic point are

$$\hat{\gamma}_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\gamma}_{xz}\ , \qquad (10)$$

$$\hat{\gamma}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\gamma}_{yz} = -\vartheta_x + \frac{\partial \hat{w}}{\partial y} - \frac{4z^2}{h^2}\frac{\hat{\tau}_{yz}}{G}, \qquad (11)$$

where

$$w'_x = \hat{w}'_x\ \ ,\ w'_y = \hat{w}'_y\ . \qquad (12)$$

We may deduce the displacements of a generic point:

$$u = \hat{u} + \vartheta_y \cdot z - \frac{4z^3}{3h^2}\frac{\hat{\tau}_{xz}}{G}\ ;\ v = \hat{v} + \vartheta_x \cdot z - \frac{4z^3}{3h^2}\frac{\hat{\tau}_{yx}}{G}, \qquad (13)$$

and from them we obtain the rotations around the x- and y-axes

$$w_{xz} = -\frac{1}{2}\left(\vartheta_y - \frac{4z^2}{h^2}\frac{\hat{\tau}_{xz}}{G} - \frac{\partial \hat{w}}{\partial x}\right);$$

$$w_{yx} = -\frac{1}{2}\left(-\vartheta_x - \frac{4z^2}{h^2}\frac{\hat{\tau}_{yz}}{G} - \frac{\partial \hat{w}}{\partial y}\right). \qquad (14)$$

Considering the third hypothesis and the value of the rotation around the z-axis at all points,

$$w_{xy} = 0 = \hat{w}_{xy} - \frac{z}{2}\left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x}\right) + \frac{2z^3}{3h^2 G}\left(\frac{\hat{\tau}_{xz}}{\partial y} - \frac{\hat{\tau}_{yx}}{\partial x}\right) = \hat{w}_{xy} - \frac{z}{2}\left(1 - \frac{4z^2}{3h^2}\right)\left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x}\right), \qquad (15)$$

we deduce

$$\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x} = 0. \qquad (16)$$

The strains ε_x , ε_{xy} , and γ_{xy} are

$$\varepsilon_x = \hat{\varepsilon}_x + \frac{\partial \vartheta_y}{\partial x}z - \frac{4z^3}{3Gh^2}\frac{\hat{\tau}_{xz}}{\partial x};$$

$$\varepsilon_y = \hat{\varepsilon}_{xy} + \frac{\partial \vartheta_x}{\partial y}z - \frac{4z^3}{3Gh^2}\frac{\hat{\tau}_{yz}}{\partial y} \qquad (17)$$

$$\gamma_{xy} = \hat{\gamma}_{xy} - \frac{\partial \vartheta_x}{\partial x}z - \frac{4z^3}{3Gh^2}\frac{\partial \hat{\tau}_{yz}}{\partial x} + \frac{\partial \vartheta_y}{\partial y}z - \frac{4z^3}{3Gh^2}\frac{\partial \hat{\tau}_{xz}}{\partial y} = \hat{\gamma}_{xy} + \left(z - \frac{4z^3}{3h^2}\right)\left(\frac{\partial \vartheta_y}{\partial y} - \frac{\partial \vartheta_x}{\partial x}\right) - \frac{8z^3}{3h^2}\frac{\partial^2 w}{\partial x \partial y} \qquad (18)$$

The shearing stresses τ_{xz} and τ_{yz} are deduced from the expressions given by Hooke's law and Lamé's equations:

$$\tau_{xz} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\tau}_{xz}\ , \qquad (19)$$

$$\tau_{yz} = \left(1 - \frac{4z^2}{h^2}\right)\hat{\tau}_{yz}\ . \qquad (20)$$

The equilibrium equations of the plate element of differential sides are

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + P = 0\ , \qquad (21)$$

$$Q_{xz} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\ , \qquad (22)$$

$$Q_{yz} = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \qquad (23)$$

The transverse shear forces in the faces of the plate element are

$$Q_{xz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz}\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(1 - \frac{4z^2}{h^2}\right)\hat{\tau}_{xz}\, dz = \frac{2h}{3}\hat{\tau}_{xz}\ , \qquad (24)$$

$$Q_{yz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz}\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(1 - \frac{4z^2}{h^2}\right)\hat{\tau}_{yz}\, dz = \frac{2h}{3}\hat{\tau}_{yz} \qquad (25)$$

and substituting in Eq.21 we obtain

$$\frac{\partial \hat{\tau}_{xz}}{\partial x} + \frac{\partial \hat{\tau}_{yz}}{\partial y} = \frac{-3P}{2h}, \qquad (26)$$

And also after substituting $\hat{\tau}_{xz}$ and $\hat{\tau}_{yz}$ with their values,

$$\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = -\frac{-3}{2Gh}P. \qquad (27)$$

The normal stress σ_z is determined by means of the following equation of the internal equilibrium of the elasticity:

$$\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0, \qquad (28)$$

from which we deduce

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} = -\left(1 - \frac{4z^2}{h^2}\right)\left(\frac{\partial \hat{\tau}_{xz}}{\partial x} + \frac{\partial \hat{\tau}_{yz}}{\partial y}\right) = = \frac{3P}{2h}\left(1 - \frac{4z^2}{h^2}\right) \qquad (29)$$

Integrating throughout the thickness between a generic point located at a height *z* and the upper surface, in which σ_z is equal to *P*, we get

$$P - \sigma_z = \frac{3P}{2h}\int_x^{\frac{h}{2}} \left(1 - \frac{4z^2}{h^2}\right)\, dz = \frac{3P}{2h}\left(\frac{h}{3} - z + \frac{4z^3}{3h^2}\right). \qquad (30)$$

Thus we deduce

$$\sigma_z = \frac{P}{2} + \frac{3Pz}{2h}\left(1 - \frac{4z^2}{3h^2}\right). \quad (31)$$

This last equation is exactly the same as that which appears in Kromm´s refined plate theory (Panc, 1975).

Now we must verify that in the lower surface, where applied loads do not exist, the following is fulfilled:

$$(\sigma_z)_{z=-\frac{h}{2}} = 0\,, \quad (32)$$

and we find that it is indeed fulfilled. The normal stresses σ_x and σ_y are deduced from the expressions given by Hooke's law and Lamé's equations:

$$\sigma_z = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right) + \frac{\mu\sigma_x}{1-\mu} = \frac{E}{1-\mu^2}\Bigg(\hat{\varepsilon}_x + \frac{\partial\vartheta_y}{\partial x}z - \frac{4z^3}{3Gh^2}\frac{\partial\hat{\tau}_{xz}}{\partial x} + \mu\left(\hat{\varepsilon}_y + \frac{\partial\vartheta_x}{\partial y}z - \frac{4z^3}{3Gh^2}\frac{\partial\hat{\tau}_{yz}}{\partial y}\right)\Bigg) + \frac{\mu\sigma_x}{1-\mu} = \frac{E}{1-\mu^2}\Bigg(\hat{\varepsilon}_x + \frac{\partial\vartheta_y}{\partial x}z - \frac{4z^3}{3Gh^2}\frac{\partial\hat{\tau}_{xz}}{\partial x} + \mu\left(\hat{\varepsilon}_y + \frac{\partial\vartheta_x}{\partial y}z - \frac{4z^3}{3Gh^2}\frac{\partial\hat{\tau}_{yz}}{\partial y}\right)\Bigg) + \frac{\mu}{1-\mu}\left(\frac{P}{2} + \frac{3Pz}{2h}\left(1 - \frac{4z^2}{3h^2}\right)\right) \quad (33)$$

Now we suppose that the in-plane normal force N_x must be null:

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz}\, dz = 0\,, \quad (34)$$

and it yields

$$\hat{\varepsilon}_x + \mu\hat{\varepsilon}_y = -\frac{\mu(1+\mu)P}{2E} \quad (35)$$

and therefore the normal stress σ_x is

$$\sigma_x = \frac{Ez}{1-\mu^2}\left(\frac{\partial\vartheta_y}{\partial x} - \mu\frac{\partial\vartheta_x}{\partial y}\right) - \frac{4z^3}{3Gh^2}\frac{E}{1-\mu^2}\left(\frac{\partial\hat{\tau}_{xz}}{\partial x} + \mu\frac{\partial\hat{\tau}_{yz}}{\partial y}\right) + \frac{3\mu Pz}{2(1-\mu)h}\left(1 - \frac{4z^2}{3h^2}\right) \quad (36)$$

Similarly, we deduce for the normal stress σ_y

$$\sigma_y = \frac{Ez}{1-\mu^2}\left(\frac{\partial\vartheta_x}{\partial y} - \mu\frac{\partial\vartheta_y}{\partial x}\right) - \frac{4z^3}{3Gh^2}\frac{E}{1-\mu^2}\left(\frac{\partial\hat{\tau}_{yz}}{\partial y} + \mu\frac{\partial\hat{\tau}_{xz}}{\partial x}\right) + \frac{3\mu Pz}{2(1-\mu)h}\left(1 - \frac{4z^2}{3h^2}\right) \quad (37)$$

with

$$\hat{\varepsilon}_y + \hat{\varepsilon}_x = -\frac{\mu(1+\mu)P}{2E} \quad (38)$$

Equations 35 and 38 allow us to verify the coherence of the presented theory, observing the fulfillment of the starting hypotheses, and to verify that the normal strains in the middle surface, $\hat{\varepsilon}_x$ and $\hat{\varepsilon}_y$, are very small and lead to $\hat{u} \approx 0\,, \hat{v} \approx 0$ and

$$\sigma_z = E\varepsilon_z - \frac{P\mu^2}{(1-\mu)} + \frac{2\mu^2}{(1-\mu)}\sigma_z \quad (39)$$

in which he third addend is small compared with σ_z

$$\sigma_z\left(1 - \frac{2\mu^2}{(1-\mu)}\sigma_z\right) = E\varepsilon_z - \frac{P\mu^2}{(1-\mu)} \quad (40)$$

and the second one,

$$\frac{P\mu^2}{(1-\mu)}, \quad (41)$$

according to Eq. 31, is small compared with the other addend of σ_z, and thus we get $\sigma_z \approx E\varepsilon_z$.

The shearing stress τ_{xy} is deduced from

$$\tau_{xy} = G\gamma_{xy} = \mathrm{G}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \quad (42)$$

Substituting, and considering the notes presented in the previous paragraph, we obtain

$$\tau_{xy} = \hat{\tau}_{xy} + Gz\left(\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x}\right) - \frac{4z^3}{3Gh^2}\left(\frac{\hat{\tau}_{xz}}{\partial y} + \frac{\hat{\tau}_{yz}}{\partial x}\right). \quad (43)$$

After integrating throughout the thickness, for example,

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z\, dz\,, \quad (44)$$

we can express the moment stress resultants by

$$M_x = D\left(\frac{\partial\vartheta_y}{\partial y} + \mu\frac{\partial\vartheta_x}{\partial x}\right) - \frac{D}{5G}\left(\frac{\partial\hat{\tau}_{xz}}{\partial x} + \mu\frac{\widehat{\partial\tau}_{yz}}{\partial y}\right) + \frac{\mu Ph^2}{10(1-\mu)}, \quad (45)$$

$$M_y = D\left(-\frac{\partial\vartheta_x}{\partial y} + \mu\frac{\partial\vartheta_y}{\partial x}\right) - \frac{D}{5G}\left(\frac{\partial\hat{\tau}_{yz}}{\partial y} + \mu\frac{\widehat{\partial\tau}_{xz}}{\partial x}\right) + \frac{\mu Ph^2}{10(1-\mu)}, \quad (46)$$

$$M_{xy} = -\frac{1-\mu}{2}D\left(\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x}\right) - \frac{(1-\mu)D}{10G}\left(\frac{\widehat{\partial\tau}_{xz}}{\partial y} + \frac{\widehat{\partial\tau}_{yz}}{\partial x}\right). \quad (47)$$

The differential equations required to determine ϑ_x, ϑ_y and $\hat{w}$ may be obtained by applying the principle of minimum energy and making null the first variation of the total potential energy or alternatively by deciding to apply variational formulation or to raise the static equilibrium equations of the plate element of differential sides. Taking the last method, we substitute the moments calculated in the last two equations

$$\Delta\vartheta_y + \frac{1+\mu}{2}\frac{\partial}{\partial y}\left(\frac{\partial\vartheta_x}{\partial x} + \frac{\partial\vartheta_y}{\partial y}\right) - \frac{1}{4}\frac{\partial}{\partial x}(\Delta\hat{w}) = \frac{5(1-\mu)}{h^2}\left(\vartheta_y + \frac{\partial\hat{w}}{\partial x}\right) - \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial x}, \quad (48)$$

$$\Delta\vartheta_x + \frac{1+\mu}{2}\frac{\partial}{\partial x}\left(\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x}\right) - \frac{1}{4}\frac{\partial}{\partial y}(\Delta\hat{w}) = \frac{5(1-\mu)}{h^2}\left(\vartheta_x + \frac{\partial\hat{w}}{\partial y}\right) + \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial y}, \quad (49)$$

which, according to Eq.16, are reduced to

$$\Delta\vartheta_y - \frac{1}{4}\frac{\partial}{\partial x}(\Delta\hat{w}) = \frac{5(1-\mu)}{h^2}\left(\vartheta_y + \frac{\partial\hat{w}}{\partial x}\right) - \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial x}, \quad (50)$$

$$\Delta\vartheta_x + \frac{1}{4}\frac{\partial}{\partial y}(\Delta\hat{w}) = \frac{5(1-\mu)}{h^2}\left(\vartheta_x - \frac{\partial\hat{w}}{\partial y}\right) + \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial y}. \quad (51)$$

These equations, along with Eq. 27, constitute a **calculation system of connected differential equations**, in the sense of simultaneously carrying out the calculation of displacements and rotations that allow us to determine ϑ_x, ϑ_y and $\hat{w}$ if we consider the boundary conditions of the plate.

III. GOVERNING EQUATIONS DISCONNECTED FOR DISPLACEMENTS AND ROTATIONS

The system formed by Eqs. 27, 51 and 52 may be disconnected, that is, to separate the calculation of displacements from the calculation of rotations. If we transform it applying Laplace's operator to Eq. 27:

$$\Delta\left(\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y}\right) + \Delta\Delta w = -\frac{3}{2Gh}\Delta P \quad (52)$$

or

$$\frac{\partial}{\partial x}\left(\Delta\vartheta_y\right) - \frac{\partial}{\partial y}\left(\Delta\vartheta_x\right) + \Delta\Delta w = -\frac{3}{2Gh}\Delta P\,. \quad (53)$$

However, Eqs.51 and 52 yield

$$\frac{\partial}{\partial x}\left(\Delta\vartheta_y\right) - \frac{\partial}{\partial y}\left(\Delta\vartheta_x\right) = \frac{1}{4}\frac{\partial^2}{\partial x^2}(\Delta w) + \frac{1}{4}\frac{\partial^2}{\partial y^2}(\Delta w) + \frac{5(1-\mu)}{h^2}\left(\frac{\partial\vartheta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{\mu h^2}{8D(1-\mu)}\Delta P\,. \quad (54)$$

According to Eq.27 the last equation can be written

$$\frac{\partial}{\partial x}\left(\Delta\vartheta_y\right) - \frac{\partial}{\partial y}\left(\Delta\vartheta_x\right) = \frac{1}{4}\Delta\Delta w - \frac{5(1-\mu)}{h^2}\frac{3}{2Gh}P - \frac{\mu h^2}{8D(1-\mu)}\Delta P, \quad (55)$$

and substituting in Eq. 53 we obtain

$$\frac{1}{4}\Delta\Delta w - \frac{5(1-\mu)}{h^2}\frac{3}{2Gh}P - \frac{\mu h^2}{8D(1-\mu)}\Delta P + \Delta\Delta w = -\frac{3}{2Gh}\Delta P. \quad (56)$$

Operating and ordering it gives

$$\Delta\Delta w = \frac{P}{D} + \frac{6(1+\mu)(\mu-2)}{5Eh}\Delta P\,. \quad (57)$$

The system of governing equations turns out to be

$$\Delta\Delta w = \frac{P}{D} + \frac{6(1+\mu)(\mu-2)}{5Eh}\Delta P \quad (58)$$

$$\Delta\vartheta_x + \frac{5(1-\mu)}{h^2}\vartheta_x = -\frac{1}{4}\frac{\partial}{\partial y}(\Delta\widehat{w}) + \frac{5(1-\mu)}{h^2}\frac{\partial\widehat{w}}{\partial y} + \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial y} \tag{59}$$

$$\Delta\vartheta_y + \frac{5(1-\mu)}{h^2}\vartheta_y = -\frac{1}{4}\frac{\partial}{\partial x}(\Delta\widehat{w}) + \frac{5(1-\mu)}{h^2}\frac{\partial\widehat{w}}{\partial x} - \frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial x} \tag{60}$$

IV. GOVERNING EQUATIONS DISCONNECTED IN DISPLACEMENTS AND "MOMENTS-SUM"

Now we propose to transform the system formed by Eqs. 5 and 10 by one disconnected in which displacements and "moments-sum," also called Marcus moment, take part.

We define "moment-sum" (designated by *M*) as,

$$\frac{M_x+M_y}{(1+\mu)} = M\left(\frac{\hat{\tau}_{xz}}{\partial x} + \frac{\hat{\tau}_{yz}}{\partial y}\right) \tag{61}$$

Adding Eq. 9 we obtain,

$$\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y} = \frac{M}{D} + \frac{1}{5G}\left(\frac{\partial\hat{\tau}_{xz}}{\partial x} + \frac{\partial\hat{\tau}_{yz}}{\partial y}\right) - \frac{12\mu P}{5Eh} \tag{62}$$

However, the sum of the derivatives of the shearing stresses in the middle surface is obtained from Eqs. 6 and 7 from which we deduce

$$\frac{\partial\hat{\tau}_{xz}}{\partial x} + \frac{\partial\hat{\tau}_{yz}}{\partial y} = G\left(\frac{\partial\vartheta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) =$$
$$G\left(\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y} + \Delta \mathrm{w}\right) \tag{63}$$

And therefore Eq. 62 is transformed into

$$4\left(\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y}\right) = 5\frac{M}{D} + \Delta \mathrm{w} - \frac{12\mu P}{5Eh} \tag{64}$$

Also, according to Eq. 26,

$$\frac{\partial\hat{\tau}_{xz}}{\partial x} + \frac{\partial\hat{\tau}_{yz}}{\partial y} = \frac{-3}{2h}P \tag{65}$$

And thus Eq. 62 gives

$$\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y} = \frac{M}{D} - \frac{3P}{10Gh} - \frac{12\mu P}{5Eh} \tag{66}$$

Now we may eliminate the difference of the derivatives of the rotations with this last equation and Eq. 27, obtaining

$$\Delta w = -\frac{M}{D} - \frac{6P}{5Gh(1+\mu)}. \tag{67}$$

On the other hand, from Eq. 54 we deduce

$$\frac{5(1-\mu)}{h^2}\left(\frac{\partial\vartheta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) = \Delta\left[\frac{\partial}{\partial x}(\vartheta_y) - \frac{\partial}{\partial y}(\vartheta_x)\right] - \frac{1}{4}\Delta\left[\frac{\partial^2}{\partial x^2}(w) + \frac{\partial^2}{\partial y^2}(w)\right] + \frac{\mu h^2}{8D(1-\mu)}\Delta P \tag{68}$$

The first member according to Eq. 27 is:

$$-\frac{5(1-\mu)}{h^2}\cdot\frac{3h^2}{12D(1-\mu)}P = \frac{-5P}{4D} \tag{69}$$

The first addend of the second member according to Eqs. 54, 59 and 60 is:

$$\frac{\Delta M}{D} - \frac{3\Delta P}{10Gh} - \frac{12\mu\Delta P}{5Eh} \tag{70}$$

and the second addend of the second member according to Eqs.59 and 60 is:

$$\frac{-\mathrm{P}}{4\mathrm{D}} - \frac{3(1+\mu)(\mu-2)}{10\mathrm{Eh}}\Delta P \tag{71}$$

Substituting in Eq. 68 we obtain

$$\frac{\Delta M}{D} = \frac{-\mathrm{P}}{\mathrm{D}} + \frac{3\Delta P}{10Gh} + \frac{12\mu\Delta P}{5Eh} + \frac{3(1+\mu)(\mu-2)}{10\mathrm{Eh}}\Delta P - \frac{3\mu}{4Gh}\Delta P \tag{72}$$

After operating and ordering , we obtain

$$\Delta M = -P - \frac{\mu h^2}{10}\frac{\Delta P}{1+\mu} \tag{73}$$

Equations 67 to 73 offer a wide range of possibilities of obtaining analytical solutions for simply supported plates.

V. DISCUSSION OF THE EQUATIONS DEDUCED

If we analyze the system of Eqs. 58 to 60, we can see that the term $-\frac{1}{4}\frac{\partial}{\partial y}(\Delta\widehat{w})$ appears as a consequence of assumption 2) of this work (parabolic distribution of the transverse shear strains through the thickness of the plate).

The terms $\frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial y}$ and $\frac{\mu h^2}{8D(1-\mu)}\frac{\partial P}{\partial x}$ are a consequence of the first assumption, which is closer to the real physical problem.

Therefore, the equations proposed are modified expressions of a third order plate theory taking into account the important condition exposed in assumption 3).

If we neglect these terms, Eqs. 58 to 60 are a close variant of the ones obtained in the Vlasov theory.

In addition, as we have pointed out, the value of the stress σ_z in this work is identical to the one proposed in the theory of plates by Kromm (1953). Similar results have been found in Reddy´s work (Reddy, 1999).

Lastly, Eq. 67 has the same structure as the one proposed by Reismann (1988), relating displacements and moment-sum, which is

$$\Delta w = -\frac{M}{D} - \frac{P}{K^2GH} \tag{74}$$

It only differs from Eq.74 in the term $(1+\mu)$.

VI. ILLUSTRATIVE EXAMPLES

Example 1: Static analysis. General Solution and study of simply supported thick plates.

If we use Fourier series to express the deflections and rotations in the form,

$$q = \sum q_n(y)\cdot sen\frac{m\pi x}{a} \tag{75}$$

$$w = \sum w_n(y)\cdot sen\frac{m\pi x}{a} \tag{76}$$

$$\theta_y = \sum T_y(y)\cdot sen\frac{m\pi x}{a} \tag{77}$$

$$\theta_x = \sum T_x(y)\cdot cos\frac{m\pi x}{a} \tag{78}$$

The general solution of the system in Eqs. 58 to 60 can be deduced by solving the following system of differential equations,

$$\left(\frac{m\pi}{a}\right)^4 w_n(y) + w''''_n(y) - 2\left(\frac{m\pi}{a}\right)^2 w''_n(y) = \frac{q_n(y)}{D} + \frac{6(1+\mu)(\mu-2)}{5Eh}\left(\left(\frac{m\pi}{a}\right)^2 q_n(y) - q''_n(y)\right) \tag{79}$$

$$-\left[\left(\frac{m\pi}{a}\right)^2 + \frac{5(1-\mu)}{h^2}\right]T_x(y) + \frac{1-\mu}{2}T''_x(y) - \frac{5(1-\mu)}{h^2}\left(\frac{m\pi}{a}\right)w_n(y) = -\frac{1}{4}\left(\left(\frac{m\pi}{a}\right)T'_x(y) + T''_y(y) + \left(\frac{m\pi}{a}\right)^2 w'_n(y) - w'''_n(y)\right) + \frac{\mu h^2}{8D(1-\mu)}q'_n(y) \tag{80}$$

$$-\frac{1-\mu}{2}\left[\left(\frac{m\pi}{a}\right)^2 + \frac{10}{h^2}\right]T_y(y) + T''_x(y) - \frac{5(1-\mu)}{h^2}w'_n(y) = -\frac{1}{4}\left(\left(\frac{m\pi}{a}\right)^2 T_x(y) + \left(\frac{m\pi}{a}\right)^3 w_n(y)\right) + \frac{\mu h^2}{8D(1-\mu)}\left(\frac{m\pi}{a}\right)q_n(y \tag{81}$$

The general solution to the system for sides *x*=0. *a*, simply supported and the sides *y*=0, *b* have arbitrary boundary conditions is

$$w = \frac{q}{24D}(x^4 - 2ax^3 + a^3x) + \frac{Pa^4}{D}\left(\sum_m \left(A_{1m} - A_{2m}\right)\frac{m\pi y}{a} ch\frac{m\pi y}{a} + (B_{1m} - B_{2m})\frac{m\pi y}{a} sh\frac{m\pi y}{a}\right) sen\frac{m\pi x}{a} \quad (82)$$

where the coefficients $A_{1m,}A_{2m}B_{1m}$ and B_{2m} have to be determined imposing the appropriate boundary conditions to the sides y=0,b.

For completely simply supported plates, the deflection w has the general form:

$$w = \frac{P}{24D}(x^4 - 2ax^3 + a^3x) + \frac{Pa^4}{D}\left(\sum_m \left(A_m ch\frac{m\pi y}{a} + B_m\frac{m\pi y}{a} sh\frac{m\pi y}{a}\right) sen\frac{m\pi x}{a}\right) \quad (83)$$

with

$$A_m = -\frac{2(\alpha_m th\alpha_m + 2)}{m^5\pi^5 ch\alpha_m}$$
$$B_m = \frac{2}{m^5\pi^5 ch\alpha_m},$$
$$\alpha_m = \frac{m\pi b}{2a} \quad (84)$$

The rotations adopt the form,

$$\theta_x = \frac{q}{24D}(4x^3 - 6ax^2 + a^3) + \frac{qa^3}{D}\left(\sum_m \left(\left[A_m(m\pi) + \frac{1}{2m^5\pi^5 ch\alpha_m}\right] ch\frac{m\pi y}{a} + B_m(m\pi)\frac{m\pi y}{a} sh\frac{m\pi y}{a}\right) cos\frac{m\pi x}{a}\right) \quad (85)$$

$$\theta_y = \frac{qa^3}{D}\left(\sum_m \left((A_m + \frac{1}{2m^5\pi^5 ch\alpha_m} + B_m)(m\pi) sh\frac{m\pi y}{a} + B_m(m\pi)\frac{m\pi y}{a} ch\frac{m\pi y}{a}\right) sen\frac{m\pi x}{a}\right) \quad (86)$$

Notice that, although the expression of the deflection w is similar to classical First Order Shear Deformation Plate Theories (FOSDPT), the rotations incorporate the terms $\frac{1}{2m^5\pi^5 ch\alpha_m}$ which are zero in FOSDPT.

Higher order plate theories, like Reddy (1990) or Shi (2007), provide similar terms to correct the expressions of the rotations.

Taking into account these expressions we can obtain, for example, the bending moments in a simply supported plate according to the theory presented, Table 1. As it can be deduced form the Table 1, the differences are significant as the ratio *a/h* decreases.

We can analyze the same problem, a simply supported isotropic rectangular plate subjected to uniformly distributed load, but using the Navier Method.

The solution form for $w(x,y)$ satisfying the boundary conditions given for a plate with all the edges simply supported is:

$$w(x,y) = c\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (87)$$

where coefficient c can be evaluated after substitution of Eq. 58

Table 1. Bending moments for different ratios *a/h* in simply supported plates

$\sum_m$	Theory	*a/h=5* M_x	*a/h=5* M_y	*a/h=10* M_x	*a/h=10* M_y
10	Present	0.0483	0.0484	0.0479	0.0479
	Reissner	0.0478	0.0479	0.0478	0.0479
15	Present	0.0483	0.0484	0.0479	0.0479
	Reissner	0.0478	0.0479	0.0478	0.0479
20	Present	0.0483	0.0484	0.0479	0.0479
	Reissner	0.0478	0.0479	0.0478	0.0479

$$c = \frac{q_0}{D\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\left[\frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} + \frac{\pi^2 t^2(2-\mu)}{10(1-\mu)}\right] \quad (88)$$

Substituting this coefficient in the displacement field (Eq. 75)

$$w(x,y) = \frac{q_0 a^4}{D\pi^4\left(1+\frac{a^2}{b^2}\right)^2}\cdot\left[1 + \frac{\pi^2 t^2(2-\mu)\left(1+\frac{a^2}{b^2}\right)}{10(1-\mu)a^2}\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (89)$$

which can be reduced to the expression obtained by Timoshenko and Woinowski (1959) in the case of t→0 and coincides exactly with the expression deduced with the higher-order plate theory of Vlasov (1958).

This author was the first to develop a consistent higher-order plate theory, establishing a third-order displacement field that satisfies the stress-free boundary conditions on the top and bottom planes of a plate (Reddy, 1990).

In addition to the advantages found for bending moments, major improvements have been found by calculating horizontal bending stresses σ_x.

In Reissner's theory, for a=b=3h and $\upsilon = 0.3$, the maximum horizontal bending stress is,

$$\sigma_x\left(x,y,-{}^{h}\!/_{2}\right) = -1{,}78\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (90)$$

The exact solution with the theory of elasticity is,

$$\sigma_x\left(x,y,-{}^{h}\!/_{2}\right) = -2{,}12\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (91)$$

In this work, , according to Eq. 36, we found

$$\sigma_x\left(x,y,-{}^{h}\!/_{2}\right) = -2{,}01\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (92)$$

reducing the error to 6%, approximately.

Example 2: Free vibration of a simply supported isotropic rectangular plate.

Applying the principle of d'Alembert to establish dynamic equilibrium equations for the study of transverse oscillations of plates, we need only consider in the equilibrium Eq. (57) the inertia forces rather than static load P.

The problem to be solved is,

$$\Delta\Delta w = -\frac{\gamma t}{D}\ddot{w} - \frac{6(1+\mu)(\mu-2)}{5E}\Delta\ddot{w} \quad (93)$$

If the deflection is expressed as,

$$w = [C_1 cos(f\tau) + C_2 sen(f\tau)]U(x,y) \quad (94)$$

where f is the frequency.

Substituting,

$$\Delta\Delta U = \frac{\gamma t}{D}f^2 U + \frac{6(1+\mu)(\mu-2)\gamma}{5E}f^2\Delta U \quad (95)$$

Boundary conditions are satisfied if we take the solution in the form,

$$U_{mn} = sen\frac{m\pi x}{a}sen\frac{n\pi y}{b} \quad (96)$$

Substituting again,

$$\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{mn\pi^2}{ab}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \frac{\gamma t}{D}f^2{}_{mn} + \frac{6(1+\mu)(\mu-2)\gamma}{5E}f^2{}_{mn}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] = 0 \quad (97)$$

and solving,

$$f_{mn} = \frac{\pi^2}{b^2}\left[n^2 + \left(m\frac{b}{a}\right)^2\right]\cdot \quad (98)$$

$$\sqrt{\frac{Et^2}{(1+\mu)\gamma}}\frac{1}{\sqrt{12(1-\mu)-1{,}2(\mu-2)\frac{t^2\pi^2}{b^2}\left[\left(m\frac{b}{a}\right)^2+n^2\right]}}$$

The first line of this equation corresponds to the classical solution (Leissa, 1973).

In order to compare the results, we consider a plate with $\frac{a}{b}=0.4$, and the paper of Liew *et al.* (1993) in which the pb-2 Rayleigh-Ritz method was adopted. These results are referred to the nondimensional frequency parameter λ given by,

$$\lambda = \frac{f^2a^2}{\pi^2}\sqrt{\frac{\rho h}{D}} \tag{99}$$

Results are shown for different ratios of thickness and length (0.01, 0.1 and 0.2) in the following Table 2. When the ratio *h/a* increases, the differences with respect to the classical theory (Reissner) are greater and the results are closer to the work of Liew (1993).

Table 2. Parameter λ of a simply supported plate. $\frac{a}{b}=0.4$, $\upsilon=0.3$

Mode	h/a=0.01	h/a=0.1	h/a=0.2
1	7.250	6.4769	5.1831
4	22.231	16.847	11.489
8	33.999	23.400	15.036
Sol.1(Liew)	7.250	6.4773	5.1831
Sol.4(Liew)	22.233	16.845	11.487
Sol.8(Liew)	33.998	23.399	15.034

VII. CONCLUSIONS

We have achieved the system formed by Eqs. 58, 59, 60, and 73, which have the same order of refinement as the one presented by Muhammad *et al.* (1990) for thick plates but with the advantage of following the exposition of the classical technical theories; it constitutes a more refined generalization than the presented one by Reismann (1988) for thick plates and that presented by Timoshenko and Woinowski (1959).

Also, assuming that the plane of application of the load is the upper plane of the plate, the first predictions of normal stress depending on the thickness values match those deduced by Kromm's theory (Kromm, 1953).

General analytical solutions to this system of equations have been presented and excellent results for simply supported plates in static and dynamic analysis have been found.

In future work we will present more complex analytical solutions for other boundary conditions, and numerical results for those cases in which analytical solutions are not possible.

REFERENCES

Baluch, M.H. and G.Z. Voyiadjis, “A refined theory for isotropic plates,” *Transactions of the Canadian Society for Mechanical Engineering*, **8**, 21-27, (1984).

Donnell, L.H., *Beams, Plates and Shells*, McGraw-Hill, New York (1976).

Ghugal, Y.M. and R.P. Shimpi, “A review of refined shear deformation theories of isotropic and anisotropic laminated plates,” *Journal of Reinforced Plastics and Composites*, **21,** 775-813 (2002).

Hencky, H., “Über die Berücksichtigung der Schubverzerrungen in ebenen Platten,” *Ing.-Arch*, **16,** 72-76 (1947).

Kienzler, R., “On consistent second-order plate theories,” *Lect. Notes Appl. Comput. Mech,*, **16**, 85-96 (2004).

Kienzler, R., H. Altenbach and I. Ott, *Theories of Plates and Shells: Critical Review and New Applications*, Springer, Berlin (2004).

Kromm, A., “Verallgemeinerte Theorie der platen-statik,” *Ing.-Arch*, **21,** 266-286 (1953).

Leissa, W., “The free vibration of rectangular plates,” *J. of Sound and Vibration*, **331**, 257-1293 (1973).

Liew, K.M., Y. Xiang and S. Kitipornchai, “Transverse vibration of thick rectangular plates — I. Compre-hensive sets of boundary conditions,” *Computers and Structures*, **49**, 1-29 (1993).

Love, A.E.H., *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, New York (1944).

Martinez Valle, J.M., “Equations transformed and expanded for a general study of isotropic plates with the Bolle-Reissner theory as a starting point,” *10th World Congr. Computational Mechanics*, Sao Paulo, Brazil (2012).

Mindlin, R.D., “Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates,” *ASME J. Appl. Mech.* **18**, 31-38 (1951).

Muhammad, A.K., M.H. Baluch and A.K. Azad, *Generalized Theory for Bending of Thick Plates. Advances in the Theory of Plates and Shells*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands (1990).

Panc, V., *Theory of Elastic Plates*, Noordhoff Int. Publishing, Leyden, The Netherlands (1975).

Reddy, J.N., “A general non-linear third-order theory of plates with moderate thickness,” *Int J Non-Linear Mech*, **25**, 677-686 (1990).

Reddy, J.N., *Theory and Analysis of Elastic Plates*, Taylor and Francis, Philadelphia, PA (1999).

Reismann, H., *Elastic Plates: Theory and Application*, John Wiley, New York (1988)

Reissner, E., “The effect of transverse shear deforma-tion on the bending of elastic plates,” *ASME J. Appl. Mech.*, **12**, 68-77 (1945).

Shi, G., “A new simple third-order shear deformation theory of plates,” *Int. J. Solids Struct.*, **44**, 4399-4417 (2007).

Soldatos, K.P., “A transverse shear deformation theory for homogeneous monoclinic plates,” *Acta Mechanica*, **94**, 195-220 (1992).

Timoshenko, S. and S. Woinowski, *The Theory of Plates and Shells*, McGraw-Hill, New York (1959).

Touratier, M., “An efficient standard plate theory,” *Int J Eng Sci.*, **29**, 901-916 (1991).

Vlasov, B.F., “Ob uravneniyakh teovii isgiba plastinok (On the equations of the theory of bending of plates)," *Izv. Akd. Nauk SSR, OTN*, **4**, 102-109 (1958).

Voyiadjis, G.Z. and D. Karamanlidis, *Advances in the Theory of Plates and Shells*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands (1990).

Voyiadjis, G.Z. and Baluch M.H., "Refined theory for flexural motions of isotropic elastic plates," *J. of Sound and Vibration*, **76**, 57-64 (1981).

Voyiadjis, G.Z. and R.W. Pecquet, "Isotropic plate elements with shear and normal strain deformations," *Intern J. for Numerical Methods in Engineering*, **24**, 1671-1695 (1987).

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