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MAXIMUM ACTIVE POWER TRANSMISSION EFFICIENCY AND POWER FACTOR DEFINITION IN HYBRID SYSTEMS WITH ARBITRARY WAVEFORMS

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***Abstract*—-- Based on a theorem giving the maximum available active power with constant transmission losses valid for arbitrary waveforms it is possible rigorously generalize the definitions of power factor based on the definition of apparent power as the maximum power available. Then it could be applied to systems with non-symmetric, non-sinusoidal, and eventually time-varying (ie non-periodic) waveforms or DC grids. By this way, the application may be extended to hybrid multi-wire systems with asymmetrical phases, with different voltages, frequencies or waveforms (i.e. non-sinusoidal waves, such as rectangular or PWM) and even DC, which can be combined eventually sharing the neutral conductor.**

***Keywords*— Apparent power, DC grids, hybrid grids, non-sinusoidal systems, power factor, power electronics.**

I. INTRODUCTION

In polyphase systems, most of the current definitions of power factor proposed in the standards (Willems *et al.*, 2005; Mohamadian and Shoulaie, 2011; Canturk *et al.*, 2015; Emanuel, 1993; 1998; 2010; Filipski, 1991; Malengret and Gaunt, 2020; Czarnecki, 1995; Pajic and Emanuel, 2006; Czarnecki and Haley, 2016; Cohen *et al.*, 1999) are based on a theorem that gives the maximum power delivered for constant transmission losses (Canturk *et al.*, 2015). However, rather than calling this theorem as "Maximum Power Theorem" we prefer to name it as "Maximum Efficiency Power Transmission Theorem" to avoid confusions with the classical theorem of electrical circuits, which states that the power that a source could supply a load becomes maximum when the load impedance is the complex complement of the Thévenin impedance.

This maximum efficiency power transmission theorem is generally proven for sinusoidal waveforms operating in the time domain using trigonometric functions, or more simply using complex phasors (Canturk *et al.*, 2015; Filipski, 1991; Czarnecki and Haley, 2016).

However, as it is shown in Tacca (2022) the theorem is also valid for non-sinusoidal waveforms, even in time-varying and no periodical systems.

This means that this theorem is valid for hybrid systems where some lines are DC supplies, other single phase sinusoidal, three phase unbalanced power subsystems and even high frequency rectangular waveform power sources (e.g. the typical 400 Hz power systems).

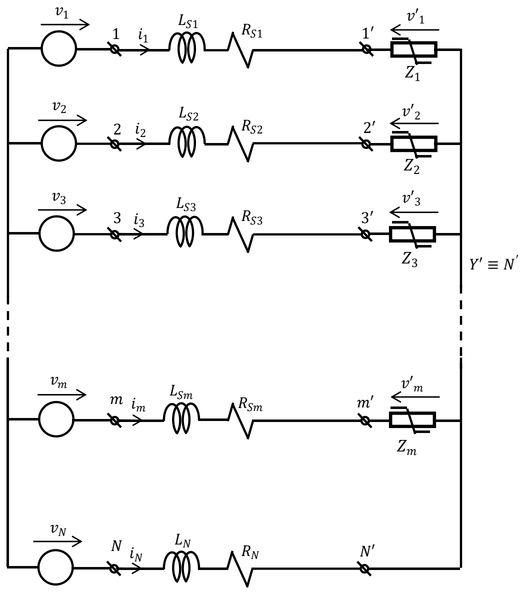

Figure 1: Circuit used for the proof of the maximum efficiency theorem (*m*: number of phases) (Tacca, 2022).

Therefore, this allows to apply the theorem to deal with problems introduced by the proliferation of power electronic loads (Willems *et al.*, 2005; Mohamadian and Shoulaie, 2011; Canturk *et al.*, 2015; Emanuel, 1993; 1998; 2010; Filipski, 1991; Malengret and Gaunt, 2020; Czarnecki, 1995; Pajic and Emanuel, 2006; Czarnecki and Haley, 2016; Cohen *et al.*, 1999; Jeon, 2005).

In this work an example of application to hybrid systems of the theorem demonstrated in Tacca (2022) is presented, and furthermore, to propose a more general definition of power factor, this concept should be extended to the DC network applications where the supply voltage is continuous but the load currents are variable (and this will be done in Section III).

II. THEOREM OF MAXIMUM EFFICIENCY

A. Determination of the maximum transmission efficiency with respect to the power source

In the circuit of Fig. 1, the waveforms are neither assumed periodical and as consequence nor sinusoidal.

It is assumed that both the resistances and the parasitic inductances of the lines can be different one from each other (Jeon, 2005), although as will be seen in the demonstration that follows, it will not be necessary to know the value of the parasitic inductances to determine the maximum possible transmission efficiency (since these do not dissipate energy).

It is assumed that the functions of the waveforms of the voltages and currents are continuous, bounded, and satisfy all the conditions required to permute the derivation and integration operations (Notice that practical waveforms have continuous derivatives bounded to values less than infinity).

The impedances of the load can be non-linear and eventually active (that is, they can contain voltage or current generators with arbitrary waveforms).

To calculate the energy transferred to the load, the instantaneous power will be integrated during a time τ corresponding to the operating time of the system, or to the chosen analysis interval (which in the particular case of periodic waves can be one or multiple periods of the network).

Let m be the number of phases, the instantaneous power delivered by the source is:

$$p = \sum_{x=1}^{m} v_x\, i_x + v_N\, i_N \tag{1}$$

where v_N and i_N are the neutral voltage and current.

Therefore, the energy consumed is:

$$W = \int_0^{\tau} p\, dt = \int_0^{\tau} \left(\sum_{x=1}^{m} v_x\, i_x + v_N\, i_N\right) dt \tag{2}$$

and the average active power is defined as:

$$P_{av} = W/\tau \tag{3}$$

An instantaneous power loss p_S will dissipate in power lines:

$$p_S = \sum_{x=1}^{m} R_{S_x}\, {i_x}^2 + R_N\, {i_N}^2 \tag{4}$$

During the time τ the energy lost in the lines will be:

$$W_S = \int_0^{\tau} p_S\, dt = \int_0^{\tau} \left(\sum_{x=1}^{m} R_{S_x}\, {i_x}^2 + R_N\, {i_N}^2\right) dt \tag{5}$$

The root mean square value of each current is defined as:

$${I_x}^2 = \frac{1}{\tau}\int_0^{\tau} {i_x}^2\, dt\ ;\ \forall x = 1, 2, 3, \dots, m, N \tag{6}$$

and that of each voltage as:

$${V_x}^2 = \frac{1}{\tau}\int_0^{\tau} {v_x}^2\, dt\ ;\ \forall x = 1, 2, 3, \dots, m, N \tag{7}$$

Also one may define:

$$v_O = R_{S_{//}} \left[\sum_{x=1}^{m} \left({}^{v_x}/_{R_{S_x}}\right) + \left({}^{v_N}/_{R_N}\right)\right] \tag{8}$$

as a reference voltage that will be zero when the phase voltages have no homopolar component and the neutral voltage is negligible. Using this reference, one may define:

$${V_{xO}}^2 = \frac{1}{\tau}\int_0^{\tau} (v_x - v_O)^2\, dt\ ;\forall x = 1, 2, 3, \dots m, N\ )$$

By using definitions (5) and (6):

$$W_S = \tau\left[\left(\sum_{x=1}^{m} R_{S_x}\, {I_x}^2\right) + R_N\, {I_N}^2\right] \tag{9}$$

The transmission efficiency is defined by:

$$\eta = (W - W_S)/W \tag{10}$$

To obtain the maximum transmission efficiency (optimal efficiency), the minimum of W_S will be found for a constant total transmitted energy W:

$$W = \tau\, P_{av} \tag{11}$$

For this purpose the Lagrange multipliers method will be applied. Using the Lagrange's method one obtains a minimum of W_S when (Tacca, 2022):

$$P_{av} = \sqrt{\left(\sum_{x=1}^{m} {V_{xO}}^2/R_{S_x}\right) + \left({V_{NO}}^2/R_N\right)}\ \sqrt{\left(\sum_{x=1}^{m} R_{S_x}\, {I_x}^2\right) + R_N\, {I_N}^2} \tag{12}$$

Then a total equivalent system voltage (or DC equivalent voltage) can be defined as:

$$V_{eq-tot} = \sqrt{R_{S_{//}}}\sqrt{\left(\sum_{x=1}^{m} {V_{xO}}^2/R_{S_x}\right) + \left({V_{NO}}^2/R_N\right)} \tag{13}$$

and a total equivalent system current (or DC equivalent current):

$$I_{eq-tot} = \sqrt{\left(\sum_{x=1}^{m} R_{S_x}\, {I_x}^2\right) + R_N\, {I_N}^2}\Big/\sqrt{R_{S_{//}}} \tag{14}$$

where:

$$R_{S_{//}} = 1/\left[\sum_{x=1}^{m}\left({}^{1}/_{R_{S_x}}\right) + \left({}^{1}/_{R_N}\right)\right] \tag{15}$$

Using both definitions, one obtains:

$$P_{av} = V_{eq-tot}\ \cdot\ I_{eq-tot} \tag{16}$$

For the constant energy W, the losses found are minimal and the transmission efficiency is maximum, so the opposite is also valid, that is, for constant losses, the transferred power calculated through (12) will be the maximum possible for these effective values of current and voltage (*rms* values with periodical waveforms).

Consequently, this maximum possible power P_{av} is the one that should be adopted as the denominator to calculate the power factor, and (12) can be adopted as the definition of apparent power for the more general case of a polyphase system discussed here.

Particular Cases of Special Interest:

There are two situations of particular interest:

1) First, when the parasitic resistances of the lines are equal to each other, but different from the resistance of the neutral conductor. That is:

$$R_{S_x} = R_S \neq R_N\ ; \forall x = 1, 2, 3, \dots, m \tag{17}$$

With this condition one gets:

$$R_{S_{//}} = 1/\left[\left({}^{m}/_{R_S}\right) + \left({}^{1}/_{R_N}\right)\right] \tag{18}$$

and defining:

$$\rho = R_N/R_S \tag{19}$$

it results:

$$R_{S_{//}} = R_S/\left[m + \left({}^{1}/_{\rho}\right)\right] \tag{20}$$

Substituting (20) in (8):

$$v_O = \left(\sum_{x=1}^{m} v_x + v_N\right)/\left[m + \left({}^{1}/_{\rho}\right)\right] \tag{21}$$

and then (12) yields:

$$P_{av} = \sqrt{\sum_{x=1}^{m} {V_{xO}}^2 + \left({V_{NO}}^2/\rho\right)}\ \sqrt{\sum_{x=1}^{m} {I_x}^2 + \rho\, {I_N}^2} \tag{22}$$

2) Case in which all the parasitic resistances of the lines are equal to each other. That is:

$$R_S = R_N \Longrightarrow \rho = 1 \tag{23}$$

and

$$R_{S_{//}} = R_S/(m + 1) \tag{24}$$

Therefore:

$$v_O = \left(\sum_{x=1}^{m} v_x + v_N\right)/(m + 1) \tag{25}$$

turns out to be the homopolar, or zero sequence, voltage of the polyphase system formed by $v_1, v_2, v_3, \dots, v_m, v_N$.

Thus, it results:

$$P_{av} = \sqrt{\sum_{x=1}^{m} {V_{xO}}^2 + {V_{NO}}^2}\ \sqrt{\sum_{x=1}^{m} {I_x}^2 + {I_N}^2} \tag{26}$$

An equivalent system voltage per phase can be defined as:

$$V_{eq} = \sqrt{\left(\sum_{x=1}^{m} {V_{xO}}^2 + {V_{NO}}^2\right)/m} \tag{27}$$

and an equivalent system current per phase:

$$I_{eq} = \sqrt{\left(\sum_{x=1}^{m} I_x{}^2 + I_N{}^2\right)/m} \quad (28)$$

which is the equivalent current originally proposed by Buchholz (Emanuel, 1998). With these definitions it results:

$$P_{av} = m\, V_{eq}\ I_{eq} \quad (29)$$

which is the maximum active power obtainable in a balanced and symmetric polyphase system with m phases operating with the given currents and effective voltages.

B. Determination of the maximum transmission efficiency with respect to the power delivered to the load

Observing Fig. 1, one may conclude that it would be more interesting and appropriate to search the maximum efficiency by minimizing transmission losses for a constant power consumed in the load, instead of imposing constant the power delivered by the source. That is, finding the minimum of W_S given by (5), keeping constant the total load energy consumed:

$$W' = \tau\, P_{av}{}' = \int_0^{\tau} p'\ dt = \int_0^{\tau} \left(\sum_{x=1}^{m} v_x{}'\ i_x\right)\ dt \quad (30)$$

where:

$$p' = \sum_{x=1}^{m} v_x{}'\ i_x \quad (31)$$

is the instantaneous power consumed by the load and

$$P_{av}{}' = W'/\tau \quad (32)$$

is the average active power consumed by the load.

Also a reference voltage is defined as:

$$v_O{}' = R_{S//} \left[\sum_{x=1}^{m} \left({v_x{}'}/{R_{S_x}}\right)\right] \quad (33)$$

and then:

$$V_{xO}'{}^2 = \frac{1}{\tau} \int_0^{\tau} (v_x{}' - v_O{}')^2\, dt\ ;\forall x = 1,2,3,\ldots m \quad (34)$$

$$V_O'{}^2 = \frac{1}{\tau} \int_0^{\tau} (v_O{}')^2\, dt \quad (35)$$

The instantaneous power losses p_S continues to be given by (4), and therefore the energy lost in transmission is also W_S given by (5).

Now, using the method of Lagrange multipliers, the minimum of W_S will be found keeping W' constant [1]:

$$P_{av}' = \sqrt{\left(\sum_{x=1}^{m} V_{xO}'{}^2/R_{S_x}\right) + \left(V_O'{}^2/R_N\right)}\ \sqrt{\left(\sum_{x=1}^{m} R_{S_x}\, I_x{}^2\right) + R_N\, I_N{}^2} \quad (36)$$

As in the previous case, a total equivalent system voltage (or DC equivalent voltage) and a total equivalent system current (or DC equivalent current) can be defined such that:

$$P_{av}' = V'_{eq-tot}\ .\ I'_{eq-tot} \quad (37)$$

where:

$$V'_{eq-tot} = \sqrt{R_{S//}}\ \sqrt{\left(\sum_{x=1}^{m} V_{xO}'{}^2/R_{S_x}\right) + \left(V_O'{}^2/R_N\right)} \quad (38)$$

$$I'_{eq-tot} = I_{eq-tot} = \sqrt{\left(\sum_{x=1}^{m} R_{S_x}\, I_x{}^2\right) + R_N\, I_N{}^2} \Big/ \sqrt{R_{S//}} \quad (39)$$

where this current is the same as that of the preceding case.

C. Comparison between both approaches

The expression of the maximum power obtained with the second approach is very similar to the obtained with the first one, which has been the preferred criterion by most international standards. The numerical results do not differ substantially between the two criteria when the neutral voltages are small and the transmission efficiencies are high. However, note that in the case of optimizing the efficiency with respect to the power transferred to the load, the reference voltage $v_O{}'$, given by (33), does not depend on the neutral voltage, which, as can be deduced from observing Fig. 1, is a voltage difficult to measure from the point of electrical energy consumption where the user is located.

For the case in which the parasitic resistances of the lines are equal to each other, but different from the resistance of the neutral conductor (condition 17), it is:

$$v_O{}' = \{m/[m + (1/\rho)]\}\ v_\Sigma' \quad (40)$$

where v_Σ' is the homopolar component of the load voltages:

$$v_\Sigma' = (1/m)\ \sum_{x=1}^{m} v_x{}' \quad (41)$$

Thus, (38) and (39) become:

$$V'_{eq-tot} = \sqrt{\left(\sum_{x=1}^{m} V_{xO}'{}^2\right) + \left(V_O'{}^2/\rho\right)} \Big/ \sqrt{m + (1/\rho)} \quad (42)$$

$$I'_{eq-tot} = \sqrt{\sum_{x=1}^{m} I_x{}^2 + \rho\, I_N{}^2}\ \sqrt{m + (1/\rho)} \quad (43)$$

This second approach based on (36) allows defining the apparent power without the need to know the neutral voltage drop as:

$$S' = \sqrt{\left(\sum_{x=1}^{m} V_{xO}'{}^2\right) + \left(V_O'{}^2/\rho\right)}\ \sqrt{\sum_{x=1}^{m} I_x{}^2 + \rho\, I_N{}^2} \quad (44)$$

which would be a simpler definition than those based on the first approach (which is used in IEEE Std. 1459).

When there is no homopolar component in the output voltages, (42) and (43) are simplified as:

$$V'_{eq-tot} = \sqrt{\left(\sum_{x=1}^{m} V_x'^2\right)/[m + (1/\rho)]} \quad (45)$$

$$I'_{eq-tot} = \sqrt{\sum_{x=1}^{m} I_x{}^2 + \rho\, I_N{}^2}\ \sqrt{m + (1/\rho)} \quad (46)$$

and (44) becomes:

$$S' = \sqrt{\left(\sum_{x=1}^{m} V_x'^2\right)}\ \sqrt{\sum_{x=1}^{m} I_x{}^2} \quad (47)$$

In both optimization approaches, losses can be expressed as:

$$P_S = \left(\sum_{x=1}^{m} R_{S_x}\, I_x{}^2\right) + R_N\, I_N{}^2 \quad (48)$$

with which, (12) and (36) become:

$$P_{av} = V_{eq-tot}\big\rfloor_{crit\#1}\ \sqrt{P_S} \Big/ \sqrt{R_{S//}} \quad (49)$$

$$P_{av}' = V'_{eq-tot}\big\rfloor_{crit\#2}\ \sqrt{P_S} \Big/ \sqrt{R_{S//}} \quad (50)$$

where $V_{eq-tot}\big\rfloor_{crit\#1}$ and $V'_{eq-tot}\big\rfloor_{crit\#2}$ are given by (13) and (38) respectively. From where the equivalent voltages per phase can be defined, according to:

$$V_{eq}\big\rfloor_{crit\#1} = V_{eq-tot}\big\rfloor_{crit\#1}/\sqrt{m} \quad (51)$$

$$V'_{eq}\big\rfloor_{crit\#2} = V'_{eq-tot}\big\rfloor_{crit\#2}/\sqrt{m} \quad (52)$$

Note that in the most general case it results:

$$V'_{eq}\big\rfloor_{crit\#1} = V_{eq}\big\rfloor_{crit\#1} - R_{S//}\ I_{eq-tot} \neq V'_{eq}\big\rfloor_{crit\#2} \quad (53)$$

which should be taken into account when proposing the equivalent circuit models.

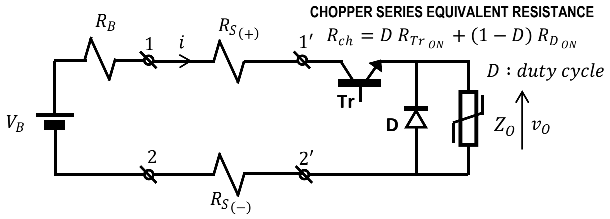

Figure 2: DC power line with a buck chopper load.

III. EXTENSION OF THE DEFINITION OF POWER FACTOR FOR APPLICATIONS IN DC GRIDS

If the definition of the power factor was generalized to include non-sinusoidal systems, it may be applied to the particular case when the supply voltage is DC, that is, with a fundamental component of zero frequency. In such a case (depicted in Fig. 2) the active power developed by an arbitrary periodical waveform current $i = i_{(t)}$ is:

$$P = \frac{1}{T} \int_0^T V_B\, i\, dt = V_B\, I_{av} \tag{54}$$

where, V_B is the DC supply voltage and $I_{av} = \frac{1}{T} \int_0^T i\, dt$ is average load current.

The current may be expressed by a Fourier series as:

$$i = I_{av} + \sum_{k=1}^{\infty} I_{k_m} \sin(k\omega t + \varphi_k) \tag{55}$$

Every harmonic component will develop an active power:

$$P_k = V_B \sum_{k=1}^{\infty} I_{k_m} \frac{1}{T} \int_0^T \sin(k\omega t + \varphi_k)\, dt = 0 \tag{56}$$

and from Parseval theorem:

$$I = \sqrt{I_{av}^{\,2} + \sum_{k=1}^{\infty} I_k^{\,2}} \geq I_{av} \tag{57}$$

The power losses in a DC transmission line with a total series parasitic resistance $R_S = R_B + R_{S_{(+)}} + R_{S_{(-)}} + R_{ch}$ are:

$$P_S = I^2 R_S \tag{58}$$

where I is the *rms* current through R_S, which is usually made up of the series association of the Thévenin resistance of the DC power supply, the series resistance of the line conductors, and the equivalent resistance of the chopper's semiconductor switches (R_{ch}).

The maximum available power is $P_{max} = V_B\, I$ and therefore the apparent power is defined as:

$$S_B = V_B\, I \tag{59}$$

then the power factor becomes:

$$PF = V_B\, I_{av} / V_B\, I = I_{av}/I \tag{60}$$

From (58) and (60) one obtains:

$$P_S = I_{av}^{\,2} R_S / PF^{\,2} \tag{61}$$

The supply power is:

$$P_B = V_B\, I_{av} \tag{62}$$

and substituting in (61):

$$P_S = P_B^{\,2} R_S / \left(V_B^{\,2}\, FP^2\right) \tag{63}$$

If P_B were constant, the losses would be increased as the power factor decreases.

Assuming that the output power P_O is constant and equal to the nominal value, for the chopper shown in Fig. 3 it is:

$$V_{O_{av}} = D\,(V_B - I_O\, R_S) \tag{64}$$

$$I_{av} = D\, I_O \tag{65}$$

$$I = \sqrt{D}\, I_O \tag{66}$$

where $D = t_{ON}/T$ is the *duty cycle factor*.

Then:

$$P_B = V_B\, I_{av} = D\, V_B\, I_O \tag{67}$$

$$P_O = V_{O_{av}}\, I_O = (V_B - I_O\, R_S)\, D\, I_O \tag{68}$$

From (60) using (65) and (66):

$$PF = \sqrt{D} \tag{69}$$

Substituting (65) in (68):

$$P_O = (V_B - I_O\, R_S)\, I_{av} \tag{70}$$

Substituting (70) in (61), yields:

$$P_S = P_O^{\,2} R_S / (V_B - I_O\, R_S)^2\, PF^2 \tag{71}$$

showing that the PF degradation increases the transmission losses. It is easy to verify that for a stated effective current I the apparent power defined by (59) must be the maximum active power that can be obtained by keeping the power losses P_S constant. With a load resistance R_O such that $I_{DC} = I = V_B/R_O$ and duty cycle $D = 1$ it results, $P_S = I_{DC}^{\,2} R_S = I^2 R_S$ and $P_{max} = V_B\, I_{DC} = V_B\, I = S$.

On the other hand, from (69) it can be concluded that $PF < 1$, so $P_B < S_B$ and a non-active power appears. As the active power is a DC one (so corresponding to a zero frequency component), there could be no reactive power and the non-active power should be considered as a distortion power $\mathcal{D}_B$:

$$\mathcal{D}_B = \sqrt{S_B^{\,2} - P_B^{\,2}} = V_B\, I_O\, D\, \sqrt{(1/D) - 1} \tag{72}$$

To improve the power factor and to reduce the power losses in the parasitic resistance R_S an input filter must be inserted.

IV. APPARENT POWER AND POWER FACTOR DEFINITION IN HYBRID SYSTEMS

A hybrid system is generally composed of several subsystems of different nature, for example, polyphase, single-phase, two-wire direct current, multi-wire direct current and others with rectangular, trapezoidal or pulse width modulated waveforms, with separate or shared neutral wires.

Adopting the criterion of defining apparent power as the maximum active power that could be obtained from the system with the given effective current and voltage conditions, it is concluded that this maximum power will be:

$$S'_{tot} = \sum_{k=1}^{n} P'_{k_{max}} = \sum_{k=1}^{n} S'_k \tag{73}$$

where n is the number of subsystems that make up the hybrid system and S'_{tot} is the total apparent power of the system that turns out to be the sum of the apparent powers of each subsystem (arithmetic apparent power) and:

$$S'_k = V'_{eq-tot_k}\; I'_{eq-tot_k} \tag{74}$$

where V'_{eq-tot_k} is the total equivalent system voltage of subsystem k given by (38) and I'_{eq-tot_k} is the total equivalent system current of the same subsystem given by (39).

Thus, the power factor of each subsystem "k" will be:

$$PF_k = P'_k / S'_k \tag{75}$$

and the total power factor of the entire hybrid system is:

$$PF_{tot} = \left(\sum_{k=1}^{n} P'_k\right) / S'_{tot} \qquad (76)$$

which can be also expressed as the weighted summation:

$$PF_{tot} = \sum_{k=1}^{n} FP_k \; (S'_k / S'_{tot}) \qquad (77)$$

V. APPLICATION EXAMPLE: POWER FACTOR CALCULATIONS IN A HYBRID SYSTEM

Figure 3 shows a hybrid system composed of three subsystems (numbered from # 1 to # 3), with different nature and topologies of direct current and alternating current. It will be shown here how to define the apparent powers of each subsystem applying a definition based on the second optimization criterion proposed in Section II, using (38) and (39) to define by (74) the apparent power of each subsystem.

It is assumed that the effective values of the voltages and currents indicated in the figures were experimentally measured, as well as the active powers of each load phase. With these data, the apparent powers and the power factors of each subsystem can be obtained and through (76) or (77), the total apparent power and the total power factor.

SUBSYSTEM #1(Fig. 3.a)

The subsystem shown in Fig. 3.a is a two-wire direct current power supply system, so the number of phases is $m = 1$.

Therefore, to apply definition (44), or also (74), one first gets from (33):

$$V_O{'}_{\#1} = \frac{1}{m+1} V'_{\#1} = \frac{1}{2} V'_{\#1} \qquad (78)$$

from which the total equivalent system effective voltage (42) is:

$$V'_{eq-tot_{\#1}} = \sqrt{\left(V'_{\#1} - V_O{'}_{\#1}\right)^2 + V_O{'}_{\#1}{}^2} = \frac{1}{\sqrt{2}} V'_{\#1} \qquad (79)$$

and the total equivalent system effective current is obtained by applying (43):

$$I_{eq-tot_{\#1}} = \sqrt{I_{\#1}{}^2 + I_N{}^2} = \sqrt{2}\, I_{\#1} \qquad (80)$$

Therefore:

$$S'_{\#1} = V'_{eq-tot_{\#1}} \; I'_{eq-tot_{\#1}} = V'_{\#1} \, I_{\#1} \qquad (81)$$

Using the first optimization criterion, the following would have been obtained:

$$S_{\#1} = V_{eq-tot_{\#1}} I_{eq-tot_{\#1}} = V_{\#1} \; I_{\#1} \qquad (82)$$

which was naturally expected according to definition (59).

Assuming negligible voltage drops in the lines:

$$V_{\#1} \cong V'_{\#1} \,.$$

Being, $V_{\#1} \cong V'_{\#1} = 500\,V$ and $I_{\#1} = 50\,A$, it is:

$$S_{\#1} \cong S'_{\#1} = 25\,kVA \,.$$

Since the load is pure resistive, it results: $FP_{\#1} = 1$.

SUBSYSTEM #2(Fig. 3.b)

In Fig. 3.b a multi-wire direct current subsystem is shown with 4 wires whose parasitic resistances are all the same. As in the previous case, it will be assumed that voltage drops in the lines can be neglected compared to the voltages in the loads:

$$V_{1_{\#2}} = 400\,V \;,\; V_{2_{\#2}} = 300\,V \;,\; V_{3_{\#2}} = 260\,V \;,$$
$$I_{1_{\#2}} = 10\,A \;,\; I_{2_{\#2}} = 20\,A \;,\; I_{3_{\#2}} = 30\,A$$

In this subsystem it is $m = 3$ and from (33):

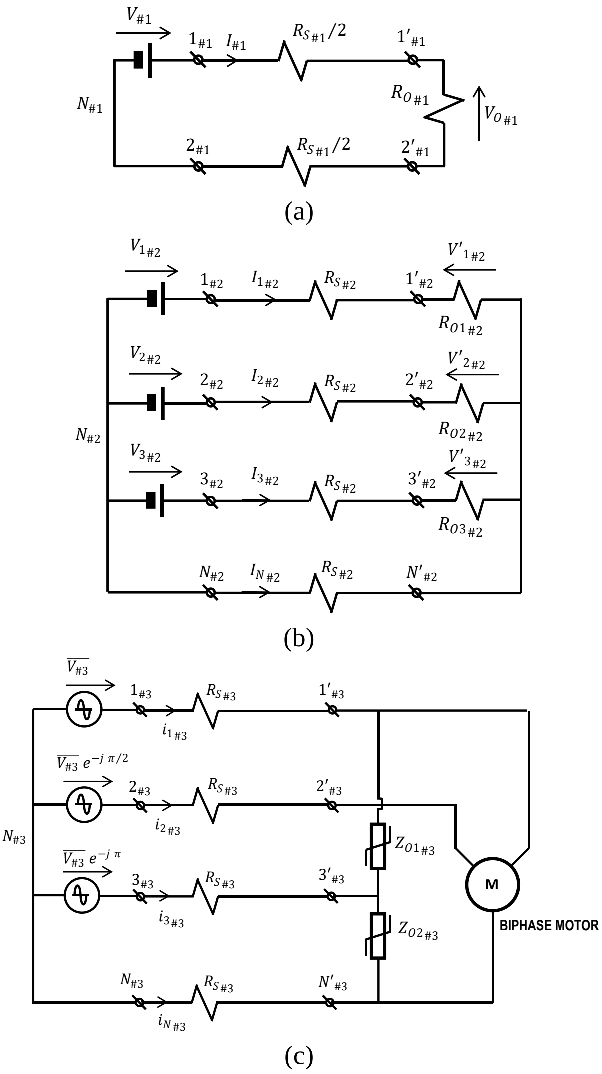

Figure 3: Hybrid system, (a) Two-wire direct-current subsystem, (b) Multiwire DC subsystem, (c) Three-wire sinusoidal subsystem, two phase in opposition plus a quadrature biphase supply.

$$V'_{O_{\#2}} = \frac{1}{4}\left(V_{1_{\#2}} + V_{2_{\#2}} + V_{3_{\#2}}\right) = 240\,V$$

According to (42) the total equivalent system voltage is:

$$V'_{eq-tot_{\#2}} = \Big(\left(V'_{1\#2} - V'_{O_{\#2}}\right)^2 + \left(V'_{2\#2} - V'_{O_{\#2}}\right)^2 + \left(V'_{3\#2} - V'_{O_{\#2}}\right)^2 + V'_{O_{\#2}}{}^2\Big)^{1/2} = 295.3\,V \qquad (83)$$

and the total equivalent system current according to (43) is:

$$I_{eq-tot_{\#2}} = I'_{eq-tot_{\#2}} = \sqrt{I_{1_{\#2}}{}^2 + I_{2_{\#2}}{}^2 + I_{3_{\#2}}{}^2 + I_{N_{\#2}}{}^2} \qquad (84)$$

where according to the law of nodes is:

$$I_{N_{\#2}}{}^2 = \left(I_{1_{\#2}} + I_{2_{\#2}} + I_{3_{\#2}}\right)^2 \qquad (85)$$

Substituting (85) in (84) and replacing by the numerical values given as data, yields:

$$I_{eq-tot_{\#2}} = I'_{eq-tot_{\#2}} = 70.7\,A \qquad (86)$$

With the values given by (83) and (86) one obtains:

$$S'_{\#2} = V'_{eq-tot_{\#2}} I_{eq-tot_{\#2}} = 20.88\,kVA \qquad (87)$$

According to data, the total active power of subsystem #2 is:

$$P_{tot_{\#2}} = V_{1_{\#2}}\, I_{1_{\#2}} + V_{2_{\#2}}\, I_{2_{\#2}} + V_{3_{\#2}}\, I_{3_{\#2}} = 17.8\ kW \quad (88)$$

and consequently, with (87) and (88), the power factor of subsystem #2 is:

$$PF_{\#2} = P_{tot_{\#2}}/S'_{\#2} = 0.852 \quad (89)$$

Note that the sum of the powers calculated as the product of each voltage and each current would also be the total apparent power considering each source and its respective load as separate subsystem from the others grouped as subsystem #2 (according to what was proposed in Section III) which would be in disagreement with the result obtained through (87). What this equation shows is that more power could be transferred to the load with the same level of transmission losses, if the multi-wire system of Fig. 3.b were replaced by the equivalent two-wire system formed by a direct voltage source $V'_{eq-tot_{\#2}}$ delivering a continuous power to the load, equal to $S'_{\#2}$.

Obviously, when establishing a standard, it should be decided the criteria to address this issue regarding DC multi-wire systems.

When the source voltages are different and are intended to supply loads independent of each other, it should be more appropriate to consider each source with its load as a separate subsystem (even if they share the neutral) and apply what was stated in Section III.

In contrast, in symmetrical sources (delivering opposite voltages: $+V_B$ and $-V_B$), either single-pair or multi-pair (such as those used in some HVDC power transmission links) it is recommended to adopt a definition of apparent power (and consequently of power factor) based on the application of the maximum transmission efficiency theorem shown here (with either of the two optimization criteria exposed) because by this way, a figure of merit would be available to show the wasted transmission capacity of the link due to load asymmetries.

SUBSYSTEM #3(Fig. 3.c)

The subsystem shown in Fig. 3.c is a polyphase sinusoidal system with opposing and quadrature supply voltages. Since the waveforms are sinusoidal, phasors can be used (see Fig. 4).

As the load impedances are not linear, the currents and voltages are not sinusoidal. However, it will be considered in the first instance that, as in the previous cases, the voltage drops in the lines may be neglected, and consequently is $V_{\#3} \cong V'_{\#3}$.

The loads represented by $Z_{O1_{\#3}}$ are electrical and electronic equipment powered by 220 V. The loads corresponding to $Z_{O2_{\#3}}$ are equipment powered by 110 V, while the motor “M” is a biphase motor powered by a quadrature system of 110 V per phase, with $V_{\#3} = 110\ Vrms$, and the *rms* currents are:

$$I_{1_{\#3}} = 8.6\ A,\ I_{2_{\#3}} = 7\ A,\ I_{3_{\#3}} = 5\ A,\ I_{N_{\#3}} = 15\ A$$

The active powers consumed in the non-linear impedances are:

$$P_{Z\_O1_{\#3}} = 500\ W,\ P_{Z\_O2_{\#3}} = 600\ W$$

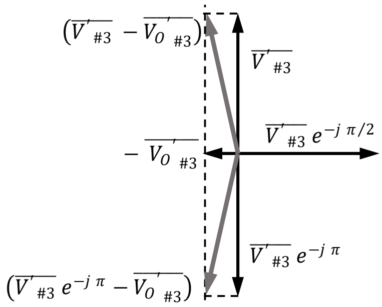

Figure 4: Phasor diagram corresponding to the opposing and quadrature biphasic hybrid subsystem of Figure 3.c.

and the active power consumed by the two-phase motor load is: $P_M = 1000\ W$.

Considering, $m = 3$, from Fig. 4 one obtains:

$$\overline{V'_{O_{\#3}}} = \tfrac{1}{4}\, \overline{V'_{\#3}}\ e^{-j\,\pi/2} \quad (90)$$

where the phasor modulus is the *rms* voltage $V_{\#3}$.

From (42) it results:

$$V'_{eq-tot_{\#3}} = \Big[\big\|\overline{V'_{\#3}} - \overline{V'_{O_{\#3}}}\big\|^2 + \big\|\overline{V'_{\#3}}\, e^{-j\,\pi} - \overline{V'_{O_{\#3}}}\big\|^2 + + \big\|\overline{V'_{\#3}}\, e^{-j\,\pi/2} - \overline{V'_{O_{\#3}}}\big\|^2 + \big\|\overline{V'_{O_{\#3}}}\big\|^2\Big]^{1/2} \quad (91)$$

From Fig. 4 it follows:

$$\big\|\overline{V'_{\#3}} - \overline{V'_{O_{\#3}}}\big\|^2 = \big\|\overline{V'_{\#3}}\, e^{-j\,\pi} - \overline{V'_{O_{\#3}}}\big\|^2 = = {V'_{\#3}}^2 + {V'_{O_{\#3}}}^2 = \left(1 + \tfrac{1}{16}\right) {V'_{\#3}}^2 \quad (92)$$

$$\big\|\overline{V'_{\#3}}\, e^{-j\,\pi/2} - \overline{V'_{O_{\#3}}}\big\|^2 = \left(V'_{\#3} - V'_{O_{\#3}}\right)^2 = \left(\tfrac{3}{4}\, V'_{\#3}\right)^2 \quad (93)$$

$$\big\|\overline{V'_{O_{\#3}}}\big\|^2 = \left(\tfrac{1}{4}\, V'_{\#3}\right)^2 \quad (94)$$

Substituting (92), (93), and (94) in (91), gives:

$$V'_{eq-tot_{\#3}} = \tfrac{1}{2}\sqrt{11}\ V'_{\#3} \cong \tfrac{1}{2}\sqrt{11}\, V_{\#3} = 182.4\ V \quad (95)$$

and from (43) comes:

$$I_{eq-tot_{\#3}} = I'_{eq-tot_{\#3}} = = \sqrt{{I_{1_{\#3}}}^2 + {I_{2_{\#3}}}^2 + {I_{3_{\#3}}}^2 + {I_{N_{\#3}}}^2} = 19.31\ A \quad (96)$$

Therefore, with (95) and (96), one obtains:

$$S'_{\#3} = V'_{eq-tot_{\#3}}\, I_{eq-tot_{\#3}} = 3.522\ kVA \quad (97)$$

The total active power consumed is:

$$P_{tot_{\#3}} = P_{Z\_O1_{\#3}} + P_{Z\_O2_{\#3}} + P_M = 2.1\ kW \quad (98)$$

and the power factor results:

$$PF_{\#3} = P_{tot_{\#3}}/S'_{\#3} = 0.596 \quad (99)$$

POWER FACTOR OF THE TOTAL SYSTEM

The total apparent power of the entire system should coincide with the maximum total active power that can be obtained from it, that is:

$$S'_{tot} = P_{tot}\rfloor_{max} = P_{\#1}\rfloor_{max} + P_{\#2}\rfloor_{max} + P_{\#3}\rfloor_{max} = S'_{\#1}\rfloor_{max} + S'_{\#2}\rfloor_{max} + S'_{\#3}\rfloor_{max} = 49.402\ kVA \quad (100)$$

and the total active power is:

$$P_{tot} = P_{\#1} + P_{\#2} + P_{\#3} = 44.9\ kW \quad (101)$$

Therefore, the total power factor is:

$$PF_{tot} = P_{tot}/S'_{tot} \cong 0.91 \quad (102)$$

VI. CONCLUSIONS AND PERSPECTIVES

The maximum efficiency theorem can be formulated following two optimization approaches, depending on the power supplied by the source or the power consumed by the load. The first approach has been preferred in the standards based on this type of definition of the apparent power obtained through demonstrations based on phasor representation (thus limiting its validity to sine waves) (Canturk *et al.*, 2015; Emanuel, 2010; Filipski, 1991). The second optimization criterion proposed here, optimizes with respect to the power consumed in the load, which leads to a definition of the apparent power a simpler that does not depend explicitly on the neutral voltage (Tacca, 2022). Perhaps, it would be better to adopt this second criterion to define the equivalent apparent power of the system, simplifying its expression so as not to need to know the proportion of power consumed in star and delta (parameter ξ of the IEEE 1459 standard) that is normally unknown (which leads to adopt $\xi = 1$ in such very common situations) (Willems *et al.*, 2005; Mohamadian and Shoulaie, 2011; Canturk *et al.*, 2015; Emanuel, 2010).

Defining an apparent power for the case of a direct current power supply allows expanding the notion of power factor for this type of electrical networks, demonstrating that when the currents are variable over time, the power factor deteriorates and must be compensate to reduce the distortion power by using power filters (passive to active).

When the users of a DC network consume energy with on-off power control or with unfiltered or partial filtered PWM, it will be convenient to define an adequate power factor for those cases, in order to evaluate the network profit.

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