OPTIMAL RECURSIVE IDENTIFICATION TECHNIQUES FROM QUANTIZED OUTPUTS

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***Abstract*— Recursive identification algorithms for the parameter estimation of linear systems from multilevel quantized outputs are introduced in this paper. The proposed algorithms are proved to be optimal in the sense that they minimize the *a posteriori* parameter estimation error covariance matrix. Numerical simulations are carried out to illustrate their performance for different quantization steps and Signal-to-Noise Ratios, as well as their capability to track time varying parameters.**

***Keywords*— Recursive Parameter Estimation, Quantized Data, Optimal Algorithms**

I. INTRODUCTION

The widespread use of networked control systems, especially for industrial control, robotics and automation, and of sensor networks, has brought a renewed interest in quantized feedback control and estimation. As pointed out in Godoy *et al.* (2011), the problems of state estimation and parameter identification are challenging since only quantized measurements are available. This situation occurs due to the use of limited capacity sensors motivated by economical or technical reasons (Pouliquen *et al.*, 2020).

The identification problem from quantized data has become known in the literature as *quantized identification* (Wang and Yin, 2007). The identification of Finite Impulse Response (FIR) models from quantized outputs has captured the attention of several research groups. For instance, in Godoy *et al.* (2011), the authors propose EM (Expectation-Maximization) type algorithms. The results are further extended to consider sparse FIR models in Godoy *et al.* (2014). In Agüero *et al.* (2007), it is shown that the accuracy of the estimates can be improved by carefully choosing the quantization mechanism. Recursive projection algorithms for the identification of FIR models with binary-valued observations are proposed and analyzed in Guo and Zhao (2013) and Guo *et al.* (2015).

The identification of ARMA (Auto Regressive Moving Average) models whose outputs are subject to finite level quantization and random packet dropouts is studied in Marelli *et al.* (2013), where the authors introduce recursive algorithms based on the maximum likelihood criterion, asymptotically achieving the minimum mean square estimation error.

Asymptotically efficient identification algorithms using quantized output observations are proposed in Wang and Yin (2007), and then extended to a scenario in which the sensor output is transmitted through a noisy communication channel. The cases of estimating FIR models as well as rational transfer function models are considered in that paper, under the rather restricted assumption of periodic inputs.

In the present paper, two recursive parameter estimation algorithms are introduced. The algorithms differ in the way the correction term used in the measurement update equation is computed. In Algorithm 1, the correction term is proportional to the difference between the quantized output and the (unquantized) prediction based on the previous parameter estimate. In Algorithm 2, instead, the correction term is proportional to the difference between the quantized output and the quantized prediction based on the previous parameter estimate. In both cases it is assumed that the unquantized output is not available for the estimation. This is in contrast to some works in the literature, see for instance (Godoy *et al.*, 2014), where the authors make use of the unquantized ouput for the parameter estimation. A comparison of the performance of the proposed algorithms with the one in (Godoy *et al.*, 2014) would not be fair, since more information is available for the estimation in the latter.

The proposed algorithms are proved to be optimal, in the sense that they minimize the *a posteriori* parameter estimation error covariance. Preliminary suboptimal versions of the algorithms were introduced by the present authors in the local conference paper (Gómez and Sad, 2020), where neither the influence of the Signal-to-Noise Ratio (SNR) nor that of the quantization step were investigated. For obvious reasons, the algorithms introduced in the present paper outperform the suboptimal versions in (Gómez and Sad, 2020).

In comparison with the work in Guo *et al.* (2015), the present approach has several advantages: (i) More general quantization schemes are considered (multilevel quantizers against binary quantizers); (ii) More general models are considered (IIR models against FIR models).

The rest of the paper is organized as follows. In Section II, the parameter estimation problem is formulated. The recursive parameter estimation algorithms are introduced in Section III. Some simulation results are presented in Section IV, and finally some conclusions are given in Section V.

II. PROBLEM FORMULATION

The system under consideration consists of an LTI (Linear Time Invariant), SISO (Single Input-Single Output), BIBO (Bounded Input-Bounded Output) stable system in

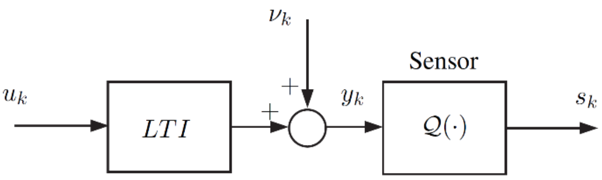

Figure 1: Schematic model.

linear regression form, whose output is corrupted by additive white Gaussian noise (AWGN), followed by a sensor whose output is subject to multilevel quantization, as depicted in Fig. 1. In that figure, $u_k \in$ P, $y_k \in$ P, $\nu_k \in$ P, and $s_k \in$ P are the input signal, the noise corrupted output of the LTI system, the output disturbance, and the quantized output of the sensor, at time k, respectively.

The system is described by the following equations

$$\boldsymbol{y_k} = \boldsymbol{\phi_k^T \theta_k} + \boldsymbol{\nu_k}, \tag{1}$$

$$\boldsymbol{s_k} = \boldsymbol{Q}(\boldsymbol{y_k}), \tag{2}$$

where $\phi_k \in \mathbb{R}^p$ and $\theta_k \in \mathbb{R}^p$, are θ_k the regressor vector and the parameter vector at time instant k, respectively, and $Q(\cdot)$ is the quantization characteristic function. To account for the possibility of having time-varying parameters, the following equation (a random walk) can be written for the parameter vector

$$\boldsymbol{\theta_{k+1}} = \boldsymbol{\theta_k} + \boldsymbol{\omega_k}, \tag{3}$$

where ω_k is the process noise at time instant k, which is assumed to be white Gaussian.

A typical quantization function is represented in Fig. 2, which corresponds to a *midtread quantizer* with uniform quantization step equal to Δ. The midtread quantization function is given by:

$$\boldsymbol{Q}(\boldsymbol{y_k}) = \boldsymbol{\Delta} \left\lfloor \frac{\boldsymbol{y_k}}{\boldsymbol{\Delta}} + \frac{\boldsymbol{1}}{\boldsymbol{2}} \right\rfloor, \tag{4}$$

where the $\lfloor \cdot \rfloor$ operator rounds downwards to the nearest integer.

***Assumption* 1.** The input u_k is bounded, and persistently exciting (Ljung, 1999).

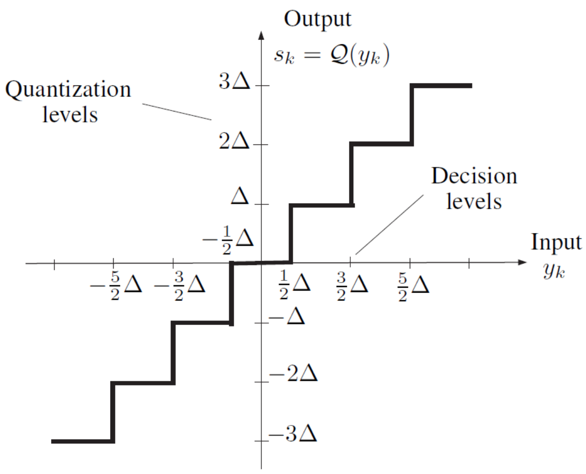

Figure 2: Characteristic function of a midtread quantizer with uniform quantization step Δ.

***Assumption* 2.** The output disturbance $\{\nu_k, k = 1,2,\cdots\}$, is an i.i.d. stochastic sequence with zero mean, finite variance R, and normal distribution: $\nu_k \sim N(0, R)$.

***Assumption* 3.** The process noise $\{\omega_k, k = 1,2,\cdots\}$, is an i.i.d. stochastic sequence with zero mean, finite variance S, and normal distribution: $\omega_k \sim N(0, S)$, and it is uncorrelated from the output disturbance ν_k.

***Assumption* 4.** It will be assumed that the regressors ϕ_k in (1) are deterministic, that is, they depend only on past inputs, and therefore they are uncorrelated from the noise ν_k. Examples of such model structures with deterministic regressors are, for instance, FIR models, and IIR models represented using orthonormal bases (Ninness and Gustafsson, 1997; Gómez, 1998).

III. PARAMETER ESTIMATION ALGORITHMS

In this section, two identification algorithms for the recursive estimation of the parameter vector θ_k of the system from input u_k and quantized output s_k measurments will be introduced. In addition, their optimality, in the sense that the covariance of the parameter estimation error is minimized, will be established. The algorithms are inspired by the Kalman Filter when applied for parameter estimation, and therefore will consist of two sets of equations: one of them projecting forward in time the current estimate and error covariance estimates to obtain the *a priori* estimates for the next time step (Time Update Equations); and the other one incorporating a new measurement into the *a priori* estimate to obtain an improved *a posteriori* estimate (Measurement Update Equations).

***Algorithm* 1**

- *Measurement Update*

$$\boldsymbol{K_k} = \boldsymbol{P_k^- \phi_k}\left(\boldsymbol{\phi_k^T P_k^- \phi_k} + \boldsymbol{R} + \boldsymbol{T}\right)^{-1}, \tag{5}$$

$$\boldsymbol{\hat{\theta}_k} = \boldsymbol{\hat{\theta}_k^-} + \boldsymbol{K_k}\left(\boldsymbol{s_k} - \boldsymbol{\phi_k^T \hat{\theta}_k^-}\right), \tag{6}$$

$$\boldsymbol{P_k} = \left(\boldsymbol{I} - \boldsymbol{K_k \phi_k^T}\right)\boldsymbol{P_k^-}. \tag{7}$$

- *Time Update*

$$\boldsymbol{P_{k+1}^-} = \boldsymbol{P_k^-} + \boldsymbol{S}, \tag{8}$$

$$\boldsymbol{\hat{\theta}_{k+1}^-} = \boldsymbol{\hat{\theta}_k^-}. \tag{9}$$

In the above equations, K_k is the Kalman gain, P_k^- is the *a priori* estimation error covariance, $\hat{\theta}_k^-$ is the *a priori* parameter estimate, P_k is the *a posteriori* estimation error covariance, $\hat{\theta}_k$ is the *a posteriori* parameter estimate, R is the output disturbance variance, S is the process noise covariance, and T is the quantization error covariance given by $T = \frac{\Delta^2}{12}$.

***Remark* 1.** Note that the correction term in the measurement update Eq. (6) depends on the difference between the quantized output and the unquantized prediction of the output based on the previous estimate.

***Proof of Optimality for Algorithm* 1**. Let $e_k = \theta_k - \hat{\theta}_k$, and $e_k^- = \theta_k - \hat{\theta}_k^-$, be the *a posteriori* and the *a priori* parameter estimation errors, respectively. From Eq. (6), it is not difficult to show that

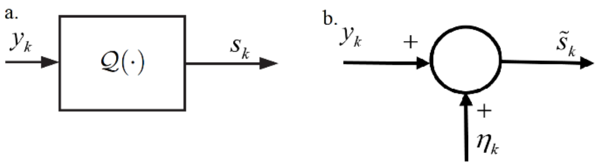

Figure 3: a) Quantization Process. b) Independent Additive Uniform Noise model.

$$
\begin{aligned}
e_k &= e_k^- - K_k\left(s_k - \phi_k^T\hat{\theta}_k^-\right)\\
&= e_k^- - K_k\left(\phi_k^T\theta_k + \nu_k + e_k^q - \phi_k^T\hat{\theta}_k^-\right)\\
&= e_k^- - K_k\left(\phi_k^T e_k^- + \nu_k + e_k^q\right)\\
&= (I - K_k\phi_k^T)e_k^- - K_k\nu_k - K_k e_k^q,
\end{aligned}
$$

where $e_k^q = s_k - \mathrm{y_k}$, is the quantization error. The a posteriori estimation error covariance becomes then

$$
\boldsymbol{P_k} = \left(\boldsymbol{I} - \boldsymbol{K_k}\boldsymbol{\phi}_k^T\right)\boldsymbol{P}_k^-\left(\boldsymbol{I} - \boldsymbol{K_k}\boldsymbol{\phi}_k^T\right)^{\mathbf{T}} + \boldsymbol{K_k}(\boldsymbol{R} + \boldsymbol{T})\boldsymbol{K}_k^T. \tag{10}
$$

The optimal gain K_k is the one that minimizes P_k, and it can be obtained by equating to zero the derivative of the trace of P_k with respect to K_k. Taking into account the properties of the derivatives of traces, it can be proved that

$$
\frac{\partial Tr\{\boldsymbol{P_k}\}}{\partial \boldsymbol{K_k}} = -2\boldsymbol{P}_k^-\boldsymbol{\phi_k} + 2\boldsymbol{K_k}\boldsymbol{\phi}_k^T\boldsymbol{P}_k^-\boldsymbol{\phi_k} + 2\boldsymbol{K_k}(\boldsymbol{R} + \boldsymbol{T}).
$$

Equating the derivative to zero results in

$$
\mathbf{0} = -2\boldsymbol{P}_k^-\boldsymbol{\phi_k} + 2\boldsymbol{K_k}\boldsymbol{\phi}_k^T\boldsymbol{P}_k^-\boldsymbol{\phi_k} + 2\boldsymbol{K_k}(\boldsymbol{R} + \boldsymbol{T}),
$$

which implies

$$
\boldsymbol{K_k}\left(\boldsymbol{\phi}_k^T\boldsymbol{P}_k^-\boldsymbol{\phi_k} + \boldsymbol{R} + \boldsymbol{T}\right) = \boldsymbol{P}_k^-\boldsymbol{\phi_k},
$$

and then

$$
\boldsymbol{K_k} = \boldsymbol{P}_k^-\boldsymbol{\phi_k}\left(\boldsymbol{\phi}_k^T\boldsymbol{P}_k^-\boldsymbol{\phi_k} + \boldsymbol{R} + \boldsymbol{T}\right)^{-1},
$$

which proves that the proposed gain in (5) is the optimal gain minimizing the a posteriori parameter estimation error covariance. □

***Remark* 2.** The algorithm considers an independent Additive Uniform Noise (AUN) model for the quantization. This means that the quantization process in Fig. 3a. can be modeled as in Fig. 3b., where η_k is additive noise with uniform distribution, and covariance $T = \frac{\Delta^2}{12}$.

Clearly, the process in Fig. 3a. is a nonlinear process, while the one in Fig. 3b. is linear. However, under the conditions of Quantizing Theorems I in Widrow and Kollàr (2008), the statistical moments of the quantized output $\boldsymbol{s_k}$ can be recovered from the moments of $\tilde{\boldsymbol{s}}_{\boldsymbol{k}}$. For this reason, the noise $\boldsymbol{\eta_k}$ is usually called Pseudo Quantization Noise (PQN).

***Remark* 3.** The Proof of Optimality assumes that the quantization error e_k^q is uncorrelated from the unquantized output y_k. This is true if the conditions of Quantizing Theorem II in Widrow and Kollàr (2008) are satisfied. For this to happen, the condition $\Delta \leq \sigma_y$ should hold, being σ_y the standard deviation of the unquantized output y_k. Note that even when $\Delta = \sigma_y$, the quantization is still very coarse.

A straightforward variation of Algorithm 1 could be obtained by considering a correction term (in the parameter update equation) proportional to the difference between the quantized output and the quantized prediction of the output based on the previous estimate (*i.e.*, the difference between two quantized signals). With this modification the following Algorithm 2 can be formulated.

***Algorithm* 2**

- *Measurement Update*

$$
\boldsymbol{K_k} = \boldsymbol{P}_k^-\boldsymbol{\phi_k}\left(\boldsymbol{\phi}_k^T\boldsymbol{P}_k^-\boldsymbol{\phi_k} + \boldsymbol{R} + 2\boldsymbol{T}\right)^{-1}, \tag{11}
$$

$$
\hat{\boldsymbol{\theta}}_{\boldsymbol{k}} = \hat{\boldsymbol{\theta}}_k^- + \boldsymbol{K_k}\left(\boldsymbol{s_k} - \boldsymbol{Q}(\boldsymbol{\phi}_k^T\hat{\boldsymbol{\theta}}_k^-)\right), \tag{12}
$$

$$
\boldsymbol{P_k} = \left(\boldsymbol{I} - \boldsymbol{K_k}\boldsymbol{\phi}_k^T\right)\boldsymbol{P}_k^-. \tag{13}
$$

- *Time Update*

$$
\boldsymbol{P}_{k+1}^- = \boldsymbol{P}_k^- + \boldsymbol{S}, \tag{14}
$$

$$
\hat{\boldsymbol{\theta}}_{k+1}^- = \hat{\boldsymbol{\theta}}_k^-. \tag{15}
$$

In the above equations, the involved variables have the same definitions as in Algorithm 1.

***Remark* 4.** Note that, in this case, the expression of the Kalman gain has to be modified to account for two quantization errors (the term $2T$ in Eq. (11)).

***Proof of Optimality for Algorithm* 2.** The proof is identical, *mutatis mutandis*, to the proof for Algorithm 1. □

IV. SIMULATION RESULTS

In this section, the performance of the parameter estimation algorithms (Algorithms 1 and 2) is evaluated through simulation, for different values of the quantization step Δ. In addition, their ability to track time-varying parameters is also analyzed. The midtread quantizer (without saturation) whose characteristic function is depicted in Fig. 2 is considered in all the simulation experiments.

Example. A second order IIR system with transfer function (in the Z-domain) given by

$$
\boldsymbol{G(z)} = \frac{\mathbf{0.4343}\boldsymbol{z}+\mathbf{1.255}}{\boldsymbol{z}^2-\mathbf{0.8}\boldsymbol{z}+\mathbf{0.15}}, \tag{16}
$$

is considered for this example. Note that the poles are $\xi_1 = 0.3$, $\xi_2 = 0.5$, implying that the system is BIBO (Bounded Input Bounded Output) stable.

The rational Orthonormal Bases with Fixed Poles (OBFP) studied in Ninness and Gustafsson (1997) are considered for the representation of the system. The bases are defined as

$$
B_l(z) = \left(\frac{\sqrt{1-|\xi_l|^2}}{z-\xi_l}\right)\prod_{i=1}^{l-1}\left(\frac{1-\bar{\xi}_i z}{z-\xi_i}\right),\ l \geq 2
$$

$$
B_1(z) = \left(\frac{\sqrt{1-|\xi_1|^2}}{z-\xi_1}\right)
$$

where ξ_l, $l = 1, 2, \cdots$, are the bases poles.

Then, by choosing the poles of the bases to match the system poles, the transfer function in (16) has a unique representation of the form

$$
\boldsymbol{G(z)} = \boldsymbol{\theta_1}\boldsymbol{B_1}(\boldsymbol{z}) + \boldsymbol{\theta_2}\boldsymbol{B_2}(\boldsymbol{z}), \tag{17}
$$

where $\theta = [\boldsymbol{\theta_1}, \boldsymbol{\theta_2}]^T = [1,2]^T$ is the nominal value of the parameter vector.

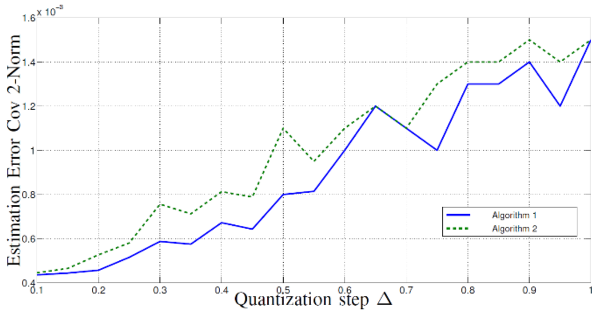

Figure 4: Parameter Estimation Error Covariance 2-Norm for Algorithm 1 (solid line) and Algorithm 2 (dashed line), as a function of the quantization step Δ.

It will be assumed that the output disturbance sequence $\{v_k, k = 1,2,\cdots\}$ is Gaussian distributed, with zero mean and standard deviation $\sigma = 0.1$.

To analyze the influence that the quantization step Δ has on the performance of the proposed recursive identification algorithms (Algorithms 1 and 2), experiments with different values of Δ within the range [0.1, 1] are performed. The system is excited with a random Gaussian signal with zero mean and variance 0.25, which is bounded, and $N = 30000$ data points are generated.

The 2-norm[1] of the parameter estimation error sample covariance matrix, as a function of the quantization step Δ is depicted in Fig. 4 for Algorithm 1 (solid line) and Algorithm 2 (dashed line). As it can be observed, Algorithm 1 outperforms Algorithm 2 for the values of the quantization step analyzed.

The top plot of Fig. 5 shows Pearson correlation coefficient (ρ) between the quantization error e_k^q and the unquantized output y_k, as a function of the quantization step Δ, while the bottom plot shows the corresponding p-values. As it can be observed, the assumption of no correlation between e_k^q and y_k is satisfied for most of the values of the quantization step being analyzed. Only for $\Delta = 0.95$ the p-value is slightly smaller than 0.05. Note that this value of Δ is close to $\sigma_y = 1.08$.

The parameter estimates for the cases of $\Delta = 0.25$, $\Delta = 0.5$ and $\Delta = 1$ are shown in Fig. 6, Fig. 7 and Fig. 8, respectively, where it can be observed that they converge to the nominal values, even for the coarsest quantization step $\Delta = 1$ (see Fig. 9, where the output y_k and the quantized output s_k are shown for the case of $\Delta = 1$). Note that to evaluate the capability of Algorithms 1 and 2 to track time-varying parameters, a step change in the parameter vector is produced at time instant k =10000, changing from $[1,2]^T$ to $[1.5,2.5]^T$. Note also that, as one would expect, the smoothest convergence is achieved for the smallest quantization step.

Figure 10 shows the sample variance for the parameter estimates $\hat{\theta}_1$ (top) and $\hat{\theta}_2$ (bottom), using Algorithms 1 and 2, as a function of the SNR[2], for $\Delta = 1$. As one would expect, the accuracy of the estimation deteriorates as the SNR decreases.

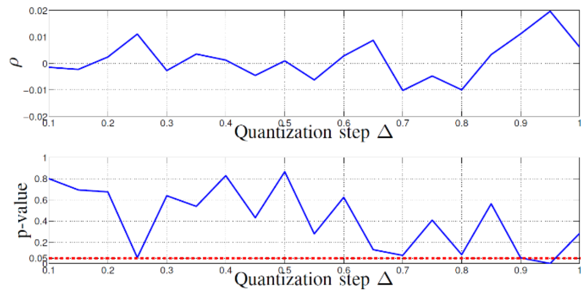

Figure 5: Top plot: Pearson correlation coefficient ρ as a function of the quantization step Δ. Bottom plot: Corresponding p-values as a function of the quantization step Δ.

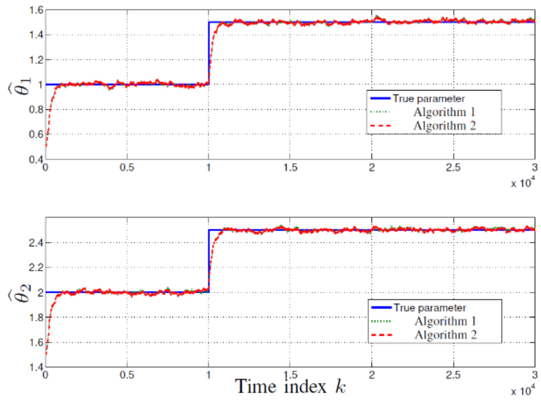

Figure 6: True parameters (solid line) and corresponding estimates for Algorithm 1 (dotted line), and for Algorithm 2 (dashed line). Top plot: parameter θ_1. Bottom plot: parameter θ_2 ($\Delta = 0.25$)

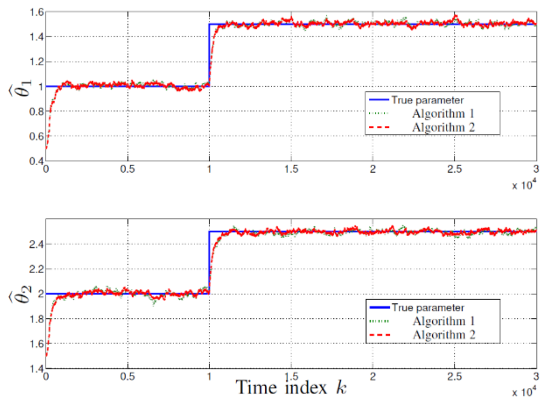

Figure 7: True parameters (solid line) and corresponding estimates for Algorithm 1 (dotted line), and for Algorithm 2 (dashed line). Top plot: parameter θ_1. Bottom plot: parameter θ_2 ($\Delta = 0.5$)

***Remark* 5**. Note that quantization noise $(s_k - y_k)$ is uniformely distributed, as can be observed from its histogram in Fig. 11, corresponding to the simulation run with $\Delta = 1$. The sample variance in this case is equal to 0.0834, while the theoretical variance of the PQN is $\frac{\Delta^2}{12} = 0.0833$, which is very close. Similar results are obtained for the quantization steps $\Delta = 0.25$ and $\Delta = 0.5$.

[1]Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. The 2-norm of the matrix is $\|A\|_2 = \sigma_1$, the largest singular value of A.

[2]The SNR is computed as: $SNR_{dB} = 10\log_{10}\left(\frac{\|\phi_k^T\theta_k\|^2}{\|v_k\|^2}\right)$

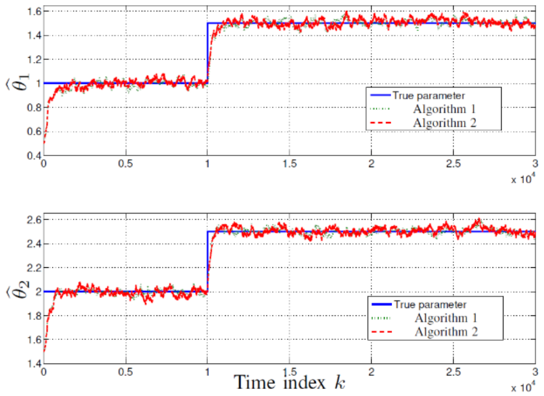

Figure 8: True parameters (blue solid line) and corresponding estimates for Algorithm 1 (green dotted line), and for Algorithm 2 (red dashed line). Top plot: parameter θ_1. Bottom plot: parameter θ_1 ($\Delta = 1$)

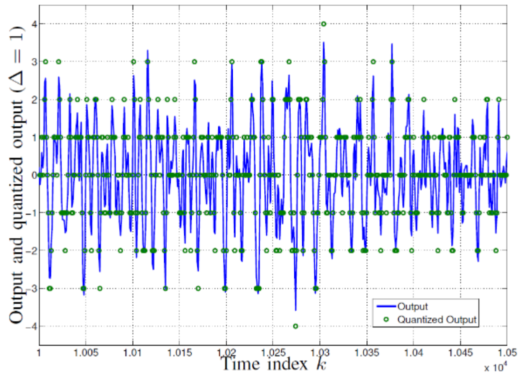

Figure 9: Output y_k (blue solid line) and corresponding quantized output s_k (green circles) for $\Delta = 1$.

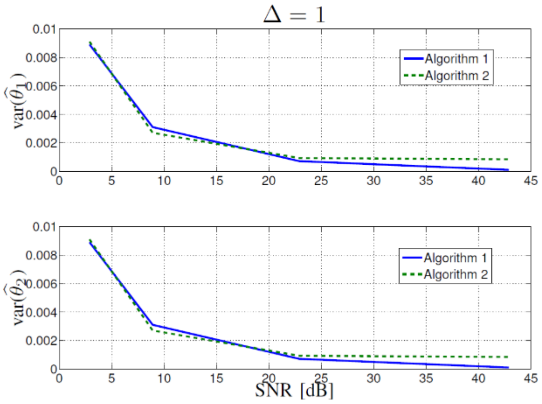

Figure 10: Sample variance of the estimates $\hat{\theta}_1$ (top) and $\hat{\theta}_2$ (bottom), using Algorithm 1 (blue solid line) and Algorithm 2 (green dashed line), as a function of the SNR, for $\Delta = 1$.

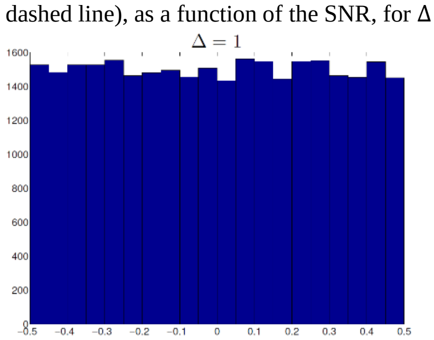

Figure 11: Histogram of the quantization error ($s_k - y_k$) for $\Delta = 1$.

V. CONCLUSIONS

In this paper, two recursive algorithms for parameter estimation of systems in linear regression form from multilevel quantized outputs have been presented. The optimality of the algorithms, in the sense that they minimize the *a posteriori* parameter estimation error covariance, has been proved. The impact of the quantization step Δ on the performance of the proposed algorithms was illustrated through simulation, as well as the influence of the signal-to-noise ratio on the accuracy of the estimation. The simulation results show that Algorithm 1 has a better perfomance than Algorithm 2. The presence of saturation in the quantizer, as well as the use of other quantization strategies, need to be further investigated. The statistical efficiency of Algorithms 1 and 2, in the sense of determining whether the estimation error covariance approaches the Cramer-Rao lower bound (Ljung, 1999; Stoica *et al.*, 2022) is being investigated by the present authors.

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