COMPARATIVE STUDY OF TWO NUMERICAL METHODS FOR SOLVING THE GRAETZ PROBLEM: THE ORTHOGONAL COLLOCATION AND CRANK-NICHOLSON METHODS

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***Abstract*–The present set of themes related to the investigations of heat transfer by convection and the transport phenomenon in a cylindrical pipe in laminar flow, commonly called the Graetz Problem, which is to explore the evolution of the temperature profile for a fluid flow in fully developed laminar flow. A numerical method was developed in this work, for visualization of the temperature profile in the fluid flow, whose strategy of calculation is based on the orthogonal collocation method followed by the finite difference method (Crank-Nicholson method). The calculations were effected through a FORTRAN computer program, and the results show that the orthogonal collocation method gives better results than the Crank-Nicholson method. The results of the present work concerning the variation of the dimensionless temperature with the axial distance for the two cases are compared with the available known results to support its accuracy.**

***Keywords*– Temperature profile, laminar flow, orthogonal collocation, Crank-Nicholson method, Convection.**

I. INTRODUCTION

The Graetz problem is known as the heat transfer problem of laminar fluid flow in ducts. It has many applications and has been studied extensively since 1885 by Graetz. The classical Graetz problem considers the forced convection heat transfer of fluids flowing in ducts while neglecting axial conduction in the fluid (Girgin and Turker, 2011).

The mathematical and approximations solutions of the Graetz problem have been obtained in numerous previous contributions. Braga *et al.* (2014) considered an extended version of the Graetz problem with axial diffusion in an infinite domain. Two heating variations were studied; first was the step change in wall temperatures and then the step change in wall heat flux. The solution is expressed in terms of an eigenseries expansion using orthogonal Sturm-Liouville eigenfunction expansion of the sought solutions. The resulting problem was then solved analytically to yield the eigenvalues and eigenvectors.

In classical the Gratez problems, the well-known dimensionless parameter Peclet number is usually assumed to be large such that the heat transport equation is simplified, as the axial diffusion becomes negligible. Lahjomri and Oubarra (1999) give the solution to the Graetz problem with axial conduction by the method of separation of variables with Dirichlet wall boundary conditions. Numerous works have been done on the classical Graetz problem. Bilir (1992) developed a profile by using the exact solution of the one-dimensional problem to represent the temperature change in the flow direction. Basu and Roy (1985) analyzed the Graetz problem for Newtonian fluid by taking account of viscous dissipation but neglecting the effect of axial conduction. Zanchini (1997) analyzed the asymptotic behavior of laminar forced convection in a circular tube, for a Newtonian fluid with constant properties. Sayed-Ahmed *et al.* (2013) studied the effects of viscous dissipation and axial conduction on the thermal entrance for a Bingham fluid in a tube. The flow is assumed to be laminar, steady state and fully developed velocity. Blackwell (1985) studied the Graetz problem for Bingham fluid inside a circular tube without viscous dissipation and axial conduction.

The major objective of this paper is to study convective heat transfer in a circular tube carrying fluid in laminar flow called commonly as the Graetz problem. A mathematical model was developed was then solved by orthogonal collocation method.

The second objective of the study is the development of a FORTRAN program to solve the problem using the orthogonal collocation method followed by the Crank-Nicholson method. This latter is based on the Lagrange polynomial approximation. The program is able to generate highly accurate results with a less instability. Surprisingly, numerical solutions to this problem generally do not give accurate results in the entrance region.The obtained results are in good agreement with the same published numerical data, and a general conclusion, which is the main thrust of this work, is summarized.

II. MATHEMATICAL MODELING

A. Flow in the conduits in laminar flow

Laminar flow in a circular conduit could be modeled as thin layers of concentric fluid hollow cylinders forming the boundary layer, each traveling at different speeds, and a homogenous single cylinder out of the boundary layer traveling at free stream velocity. Based on this model, the heat transfer between adjacent fluids cylinders are as follows:

1. In directions perpendicular to the cylinder surface, heat transfer occurs only through conduction.
2. In the direction of the flow, the heat transfer occurs through convection and conduction simultaneously.

In most cases, the hollow fluid cylinder is thin enough such that conduction could be neglected.

The flow taking place in the z direction has a cylindrical symmetry compared to the radius vector r. The speed is maximum at the axis of the control volume:

$$u = u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{1}$$

The average speed of the control volume could be calculated based on the volume flow (Ower and Pankhurst, 1974):

$$\bar{u} = 2u_{\max} \tag{2}$$

This shows that the speed profile is parabolic according to the normal variable space of flow and the maximum speed is equal twice the average speed.

B. The heat equation in cylindrical coordinates

The general equation for heat transfer in cylindrical coordinates developed by Bird *et al.* (1960) is as follows:

$$u_z\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\nabla^2 T \tag{3}$$

$$\rho C_p\left(\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]+$$
$$2\mu\left\{\left(\frac{\partial u_r}{\partial r}\right)^2+\left[\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta}+u\right)\right]^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right\}$$
$$+\mu\left\{\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2+\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2\right\}+\mu\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2 \tag{4}$$

Considering that the flow is steady, the fully developed laminar flow (Re < 2400), and if the thermal equilibrium had already been established in the flow, then $\partial T/\partial t=0$. The dissipation of energy would also be negligible. Other physical properties would also be constant and would not vary with temperature.

After applying the above assumptions, Eq. (4) reduces to the following form:

$$u_z\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{5}$$

Given that the flow is a fully developed laminar flow, then the velocity profile would have followed the parabolic distribution across the pipe section, represented by

$$u = \bar{u}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{6}$$

By replacing the speed term in Eq. (5), we get

$$\bar{u}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{7}$$

Solving the equations requires the boundary conditions as set in Fig. 1, thus

$$\begin{gathered} @\, z=0, T=T_0 \\ @\, r=0, \partial T/\partial r = 0 \\ @\, r=R, T=T_\omega \end{gathered} \tag{8}$$

It is more practical to study the problem with standardized variables from 0 to 1. To do this, new variables without dimension (known as adimensional) are introduced and defined as $\theta=(T_\omega-T)/(T_\omega-T_0)$, $x=r/R$ and $y=z/L$.

After making the necessary arrangements and simplifications, the following simplified equation is obtained:

$$\left(1-x^2\right)\frac{\partial \theta}{\partial y} = \frac{k}{\rho C_p 2\bar{u}R^2}\left[\frac{1}{x}\frac{\partial \theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\right] \tag{9}$$

where the term $2\bar{u}R\rho C_p/k$ is the adimensional number known as Peclet number (Pe), which in fact is the Reynolds number divided by Prandtl number. In a steady state condition, the partial differential equation resulting from this adimensional form can be written as follows:

$$\left(1-x^2\right)\frac{\partial \theta}{\partial y} = \frac{L}{PeR}\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right)\right] \tag{10}$$

This equation, if subjected to the new boundary conditions, would be transformed to the followings:

$$@\, z=0,\ T=T_0\ @\ y=0,\ \theta=1, ,$$
$$@\, r=0,\ \frac{\partial\theta}{\partial r}=0 \Rightarrow @\, x=0,\ \frac{\partial \theta}{\partial x}=0 \Rightarrow \theta(0,y)\neq 0\ ,$$
$$@\, r=R,\ T=T_\omega \Rightarrow @\, x=1,\ \theta=0 \Rightarrow \theta(1,y)=0$$

To solve Eq. (10), it is hereby proposed that the orthogonal collocation method could be applied, based on orthogonal polynomials.

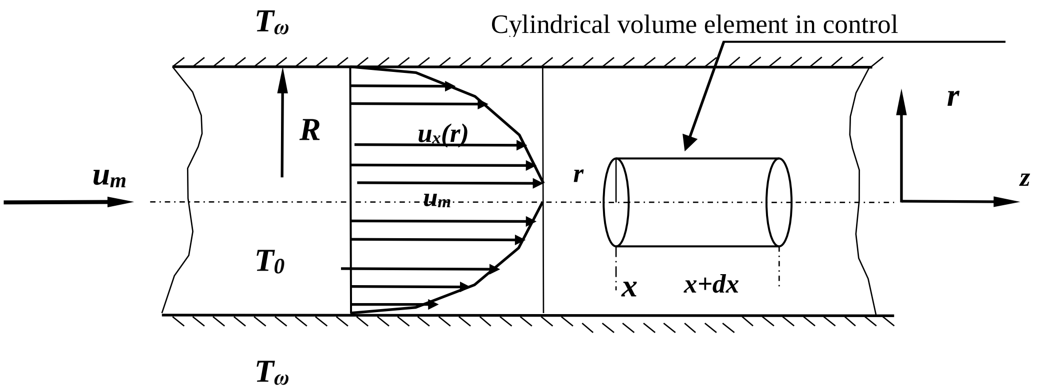

Figure 1: Geometry of a laminar flow in control.

C. Application of the orthogonal collocation method

For this problem, the governing Eq. (10) is subject to the following conditions:

CI : @ y=0, θ=1

CL1 : @ x=0, $\partial\theta/\partial x$

CL2 : @ x=1, θ=0

Operators' gradient and Laplacian of the functions are given by Babu and Sastry (1998) and Villadsen and Michelsen (1978):

$$\left(\left.\frac{\partial\theta}{\partial x}\right|_{x=x_i}\right)=\sum_{j=1}^{n+1}A_{i,j}\,\theta(x_j,y) \tag{11}$$

$$\nabla^2\theta\Big|_{x_i}=\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right)\Bigg|_{x_i}\right)=\sum_{j=1}^{n+1}B_{i,j}\,\theta(x_j,y) \tag{12}$$

To evaluate the points of the interior collocation and the two points at the end, the substitution of Eq. (10) in the governing partial differential equation with the initial and boundary conditions leads us to

$$\left|\frac{\partial\theta}{\partial y}\right|_{x_i}=\frac{L}{PeR\left(1-x_i^2\right)}\left[\sum_{j=1}^{n+1}B_{i,j}\,\theta(x_j,y)\right] \tag{13}$$

$$\sum_{j=1}^{n+1}A_{i,j}\,\theta(x_j,y)=0 \text{ and } \theta(1,y)=0 \tag{14}$$

A partial expansion of Eq. (13) gives

$$\left|\frac{\partial\theta}{\partial y}\right|_{x_i}=\frac{L}{PeR\left(1-x_i^2\right)}\left[B_{i,1}\theta(0,y)+\sum_{j=2}^{n}B_{i,j}\,\theta(x_j,y)+B_{i,n+1}\theta(1,y)\right] \tag{15}$$

Note that in the above expression (Stewart *et al.*, 1985, Villadsen, 1970), we must use some other means to eliminate. That can be done by developing Eq. (14) of the boundary condition.

$$A_{1,1}\theta(0,y)+\sum_{j=2}^{n}A_{1,j}\,\theta(x_j,y)+A_{1,n+1}\theta(1,y)=0$$

Solving for, we get (Michelsen, 1994):

$$\theta(0,y)=-\frac{1}{A_{1,1}}\left[\sum_{j=2}^{n}A_{1,j}\,\theta(x_j,y)\right] \tag{16}$$

And finally, substituting back the governing equation and after rearranging, the following equation is obtained:

$$\left|\frac{\partial\theta}{\partial y}\right|_{x_i}=\frac{L}{PeR\left(1-x_i^2\right)}\left\{\sum_{j=2}^{n}B_{i,j}\,\theta(x_j,y)-\left[\frac{B_{i,1}}{A_{1,1}}\right]\sum_{j=2}^{n}A_{1,j}\,\theta(x_j,y)\right\} \tag{17}$$

It is noticed that the orthogonal collocation for infinitesimal x and y for the partial differential equation (PDE), i.e., formation of a PDE to a series of ODE (ordinary differential equation). To solve this series of ODE, the traditional Runge-Kutta method of order six should be used. With this intention, two subroutines FUN and DFUN are needed to be able to apply the algorithm STIFF3, which are detailed in the next part.

III. NUMERICAL SOLUTION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

A numerical solution procedure is proposed using the orthogonal collocation method, which is based on an approximation of Lagrange polynomials. It is able to deliver highly accurate results with stiffness and reduced instability. This method reduces the degree of the equation and can be solved in a step-by-step manner.

In the following, subroutines that are used in the simulation, which require the determination of several input and output parameters are presented. The language used in the main program is FORTRAN which itself contains other subroutines.

A. The procedure of the numerical solution

By applying the equations developed to the problem, a more general equation could be written as follows:

$$\left[\frac{d\theta}{dy}\right]_{x_i}=\frac{L}{PeR\left(1-x_i^2\right)}\left\{\sum_{j=2}^{n}B_{i,j}\,\theta(x_j,y)-\left[\frac{B_{i,1}}{A_{1,1}}\right]\sum_{j=2}^{n}A_{1,j}\,\theta(x_j,y)\right\} \tag{18}$$

Now the problem is reduced to solving a system of simultaneous equations in view of obtaining θ at each point of the collocation point. It starts from the normal form at this point to fully develop the program needed to solve this basic example, such that for the latter example, only the discretization of the equation is given. In the example, the Lagrange polynomial of the order six was used (that is to say that we have six inner roots or collocation points), and both the temperature boundary conditions for θ variable were taken into account. For economy of computation time and programming, it is recommended to resolve the six internal temperatures θ variable, instead of all the eight points (eight equations). Thus, the optimal procedure was to write three short routines FUN, DFUN, and OUT (Stanley, 1984).

The main program is written in FORTRAN to solve the above equations. This program communicates with other subroutines to perform the calculations needed to solve different parameters. This program calculates the non-dimensional temperature profiles in the tube with a fully developed laminar flow.

IV. APPLICATION OF THE CRANK-NICHOLSON METHOD

Therefore, for this problem, there is a non-linear PDEs with the following boundary conditions:

$$\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}=\left(1-R^2\right)\frac{\partial U}{\partial X} \tag{19}$$

B.C.1. $U=1, X=0$,

B.C.2. $\partial U/\partial R=0, R=0$

B.C.3 $U=0, R=1$

A numerical method can be used to solve Eq. (19). The analog finite difference corrected the second order; Crank-Nicholson method (Rosenberg, 1969) is used in the following manner:

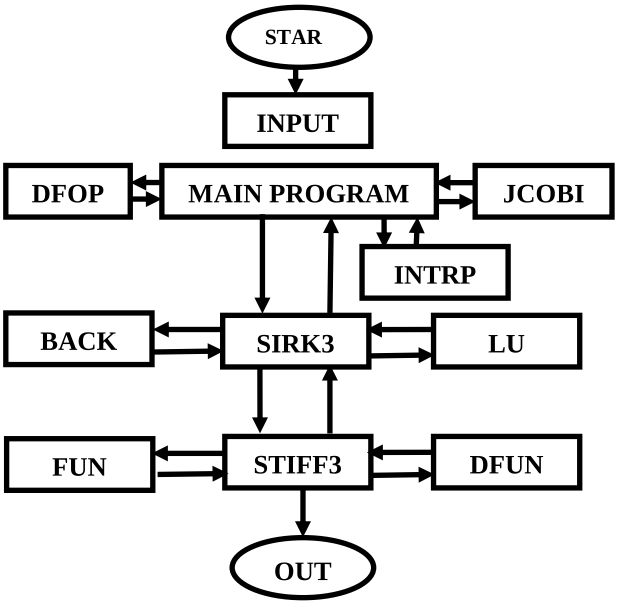

Figure 2: Digital simulation a flow chart.

$$\left.\frac{\partial^2 U}{\partial R^2}\right|_{m,n+\frac{1}{2}} = \frac{1}{2}\left(\frac{U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1}}{h^2} + \frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{h^2}\right)$$

$$\left.\frac{1}{R}\frac{\partial U}{\partial R}\right|_{m,n+\frac{1}{2}} = \frac{1}{2R_m}\left(\frac{U_{m+1,n+1} - U_{m-1,n+1}}{h} + \frac{U_{m+1,n} - U_{m-1,n}}{h}\right) \quad (20)$$

$$\left.\left(1-R^2\right)\frac{\partial U}{\partial X}\right|_{m,n+\frac{1}{2}} = \left(1-R_m^{\ 2}\right)\left(\frac{U_{m+1,n} - U_{m,n}}{2(k/2)}\right)$$

where h and k are the mesh sizes of R and X respectively. The Crank-Nicholson equation is the result of successively application of forward and backward difference equations. This is stable for all values of the ratio h/k. After the derivatives are replaced, Eq. (19) becomes

$$A_m U_{m-1,n+1} + B_m U_{m,n+1} + C_m U_{m+1,n+1} = D_m \, ,2 \le m \le M \quad (21)$$

where $A_m = \frac{1}{2h^2} - \frac{1}{4R_m h}$, $B_m = -\frac{1}{h^2} - \left(1-R_m^2\right)\frac{1}{k}$, $C_m = \frac{1}{2h^2} + \frac{1}{4R_m h}$ and $D_m = \left(\frac{1}{2h^2} - \frac{1}{4R_m h}\right)U_{m-1,n+1} + \left(\frac{1}{h^2} - \left(1-R_m^2\right)\frac{1}{k}\right)U_{m,n+1} - \left(\frac{1}{2h^2} + \frac{1}{4R_m h}\right)U_{m+1,n+1}$.

Equation (21) contains three unknowns: $U_{m-1,n+1}$, $U_{m,n+1}$ and $U_{m+1,n+1}$. Analogs of finite difference B.C.2 and B.C.3 are:

$$\text{B.C.2: } B_1 U_{1,n+1} + C_1 U_{2,n+1} = D_1 \quad (22)$$

where $B_1 = 1/k + 2/h^2$, $C_1 = -2/h^2$ and $D_1 = \left(1/k - 2/h^2\right)U_{1,n} + \left(2/h^2\right)U_{2,n}$, and

$$\text{B.C.3: } A_M U_{M-1,n+1} + B_M U_{M,n+1} = D_M \quad (23)$$

where $A_M = 1/2h^2 - 1/4h$, $B_M = -1/h^2$ and $D_M = \left(1/4h - 1/2h^2\right)U_{M-1,n} + \left(1/h^2\right)U_{M,n}$.

The Eqs. (21-23) are an M+1 system of algebraic equations with M+1 unknowns. Using a known auxiliary condition, which is CL1: $U_{m,1}=1$ at $X=0$.

The system of the equations above can be solved to yield $U_{1,2}$, $U_{2,2}$, $U_{3,2}$, …, $U_{M,2}$. These values are in turn used to calculate $U_{1,3}$, $U_{2,3}$, $U_{3,3}$, …, $U_{M,3}$. This system of simultaneous equations can be represented by a coefficient of tridiagonal matrix and can be solved more effectively by the method of Thomas (Rosenberg, 1969). The algorithm of the method first calculates:

$$\beta_m = B_m - A_m C_{m-1}/\beta_{m-1}$$

with $\beta_1 = B_1$, and

$$\gamma_m = \frac{D_m - A_m \gamma_{m-1}}{\beta_m}$$

with $\gamma_1 = \frac{D_1}{\beta_1}$.

U values are then calculated back from U_M to U_1

$$U_{M,n+1} = \gamma_M$$

$$U_{m,n+1} = \gamma_M - \frac{C_m U_{m+1,n+1}}{\beta_m}$$

V. RESULTS AND DISCUSSION

Figures 3 and 4 illustrate the results obtained by the method of orthogonal collocation in a cylindrical model. These results are expressed in different points of internal collocation n. Therefore, more number of nodes used, the better the results of the acuity. We can see from this figure that at most n, the higher normalized dimensionless temperature evolution of the unit decreases exponentially to zero. The value of θ=1 corresponds to the fluid temperature at the inlet, while θ= 0 corresponds to the wall temperature.

Figure 5 presents the non-dimensional θ distribution as a function of Z at different values of Peclet number, Pe. The figure shows the influence of the Peclet number

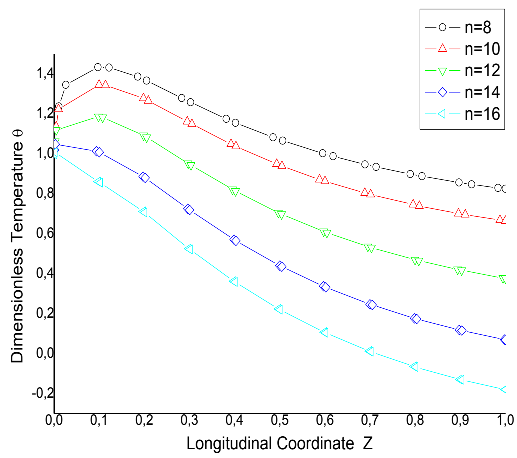

Figure 3: Dimensionless temperature, θ profile as a function of Z at various collocation points n.

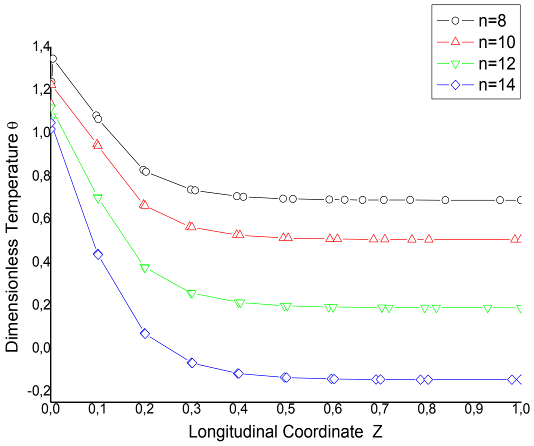

Figure 4: Temperature profile θ according to Z at Pe=0.5, R/L = 0.1.

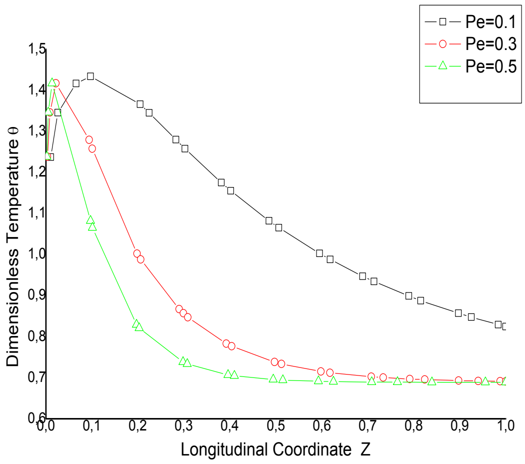

Figure 4: The distribution of the non-dimensional temperature θ with Z at different Peclet number.

in the flow regime. It must be remembered that the Peclet number is defined as a measure of the ratio of the power transmission rate by convection to the energy transport rate by conduction. It is observed that the variation is large when the Pe value increases.

Figure 6 shows the temperature variations at specific distances R/L from the intake. The numerical solution shows that the iso-temperature lines are almost linear, i.e., the temperature remains almost constant along a streamline far from the intake. We see that the results for R/L=0.1 are the above results when R/L=0.01 and 0.001. In other words, as the distance from the entrance of the tube reduces (R/L reducing), the wall temperature value gradually becomes predominant in the liquid stream.

Figure 7 shows the distribution of the non-dimensional temperature U with Z. The curves were obtained by the implicit Crank-Nicholson method with different number of mesh. It should be noted that for low values of the number of mesh N in the discretization, the influence of N is more significant to the distribution of temperature compared to when N is high.

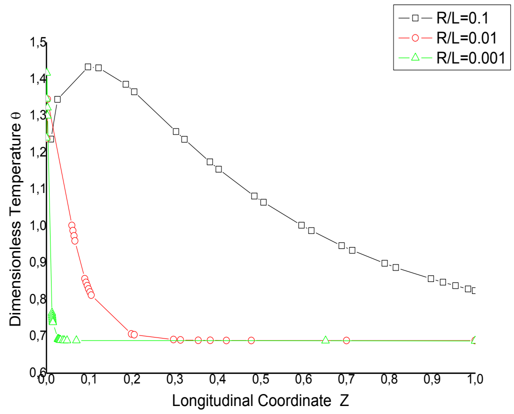

Figure 6: The distribution of the non-dimensional temperatures θ with Z, taking account the influence of the ratio R/L.

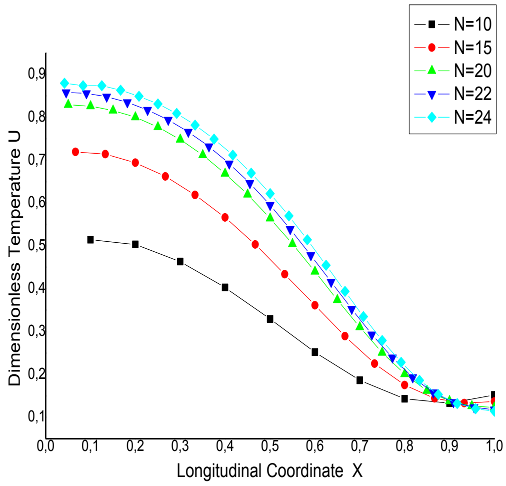

Figure 7: Variation in the non-dimensional temperature U in X according to the diagram of Crank-Nicholson.

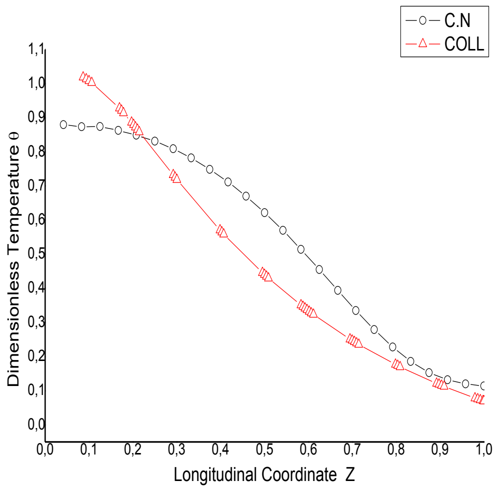

Figure 8: Comparison of orthogonal collocation (n=29) with Crank-Nicholson method (N=30).

Figure 8 compares the computation results by the orthogonal collocation method and that of the implicit Crank-Nicholson method. It can be seen that the two curves deviate significantly when the number of collocations or the number of point of mesh are large. This means that the technique of orthogonal collocation allowed fast simulations whereas the implicit Crank-Nicholson method would be more reliable even if it requires more intensive computer time.

In order to compare the present results with the known results of the previous works. We found the work of Blackwell (1985) which appears to be more accurate than those available in the literature. To confront the results from our model, we chose to present the results of numerical distribution of temperature with the method of orthogonal collocation which gives the best results. Figure 9 plots the comparison results. It has been found that present results of the collocation method are in good agreement with the results of Blackwell.

VI. CONCLUSIONS

This paper shows how temperature profile of a fully developed laminar flow in a hot wall tube could be determined. Firstly, the equation was simplified. Assuming the flow is incompressible, it is possible to decouple the thermal and dynamic equations. Then the dimensionless number, the Peclet number, was introduced. Its influence in the case of a flow in a tube (Graetz problem) was examined. T his is the basics of all internal thermal flow problems. This work consists of solving the Graetz problem by two different methods; however, it was showed that each has their own merit.

From this numerical study, we reached the following conclusions:

- The dimensionless temperature is very high at the beginning of the entrance region of tube and thereafter decreases exponentially to zero
- For fully developed laminar conditions and for large Peclet numbers, dimensionless temperature goes to a stable value for constant wall temperature
- The orthogonal collocation method gives better results than the Crank-Nicholson method

The numerical data obtained in this work, also supported by the literature results; show that the curvature of the wall tube represents an efficient solution for enhancing the heat transfer in the case of laminar flow regime.

This study also shows that it would be possible to guide the selection of suitable numerical simulation procedures when dealing with a complex problem of convection. This work also showed the solution to a form of heat transfer in a fairly simple problem of a laminar pipe flow. This problem is a fundamental problem, because it is the basis of all possible problems of heat transfer in a pipe flow.

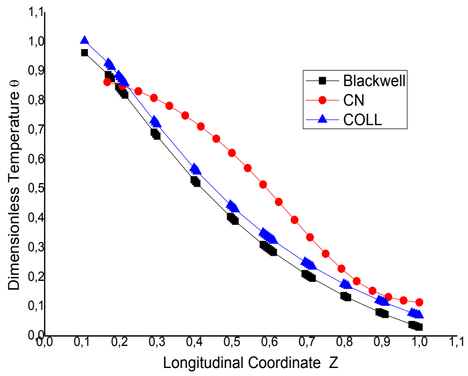

Figure 9: A comparison between the present numerical results with the Blackwell results.

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