A SIMPLE APPROACH TO ANALYZE THE FULLY DEVELOPED TWO-PHASE MAGNETOCONVECTION TYPE FLOWS IN INCLINED PARALLEL-PLATE CHANNELS

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***Abstract*— A numerical approach is presented herein to analyze steady-state, fully developed MHD-type stratified two-fluid flow with heat transfer in inclined parallel-plate channels. The one dimensional model is extended to include several effects, such as buoyancy, moving plates, porous layers, inclined applied magnetic field, Joule and viscous heating and volumetric heat generation/absorption. Evaluation includes axial induced magnetic field, although its effect on the flow is negligible. The plates are maintained at constant temperatures, and the thermophysical properties are considered to be constant. The governing non-linear and coupled equations are solved through a well-established numerical routine to solve stiff two-point boundary value problems. The results are presented and validated for a large range of governing parameters and situations.**

***Keywords*— Magnetoconvection, MHD, two-phase flow, stratified flow, numerical solution.**

I. INTRODUCTION

In recent decades, there has been a significant increase in the number of studies on electrically conducting fluid flows under the effect of magnetic fields in ducts and enclosures (Magnetohydrodynamics, MHD). This increase was mainly motivated by important industrial applications in metallurgy, MHD power generators and nuclear engineering (Davidson, 2001). Several MHD single-phase flow and heat transfer studies have been extensively presented and have covered different physical parameters. Due to the interest in reducing pumping power requirements, the stratified flow of liquids has been theoretically and experimentally studied. For liquid metals, the power requirement is further reduced. Most existing studies have been limited to MHD two-fluid flow and heat transfer in horizontal and vertical channels. Malashetty and Leela (1992) analyzed, analytically, the two-phase fully-developed MHD forced convection in a non-porous horizontal channel with stationary plates at the same temperature. Zivojin *et al.* (2010) and Nikodijevic *et al*. (2011) analyzed the analytical effect of an inclined magnetic field on flow between two horizontal moving plates. Umavathi *et al.* (2005) and Malashetty *et al.* (2006) analyzed a vertical configuration by the perturbation method. Both works considered a magnetic field normal to gravity, and included heat generation/absorption terms to take into account the presence of exothermic and/or endothermic chemical reactions.

As some applications are closer to inclined geometries, several works on this specific configuration were reported along the years, embodying various physical effects. Malashetty and Umavathi (1997) applied the approximate perturbation method to study the two-phase fully-developed mixed convection in an inclined channel, where only one phase is electrically conducting. Umavathi *et al.* (2010) analyzed the same geometry, but considered the influence of a moving plate on flow and heat transfer characteristics. By considering both fluids electrically conducting, Malashetty *et al.* (2001) extended the previous work of Malashetty and Umavathi (1997). Murty and Prakash (2014) evaluated the influence of rotation on the two-phase MHD mixed convection in an inclined channel.

Although analytical solutions can be established for simple situations, the equations become coupled and non-linear when more physical effects are considered. Thus, only approximate analytical or purely numerical (finite-difference or finite-element) methods have been used in the literature. Therefore, this manuscript aims to illustrate an efficient and robust approach to solve one-dimensional fully developed two-fluid immiscible MHD flow with heat transfer in inclined channels. Several other physical aspects (electrically conducting or non-conducting fluids; horizontal, vertical, and inclined channels; rotating system, porous layers; heat generation or absorption; forced, free or mixed convection, dispersion, etc.) can also be incorporated into the mathematical model. Some of these aspects were analyzed herein, accompanied by a critical comparison with literature results.

II. METHODS

A. Mathematical Formulation

The geometry of the problem is depicted in Fig. 1, consisting of two parallel plates, which are infinitely extended in the x^* and z^* directions, and inclined by an angle ϕ to the horizontal. Regions $0 \le y^* \le h_1$ and $-h_2 \le y^* \le 0$ are named regions I and II, respectively. The plates are held at constant temperatures, $T_{w1} \ge T_{w2}$, and are allowed to move with constant velocities, U_{w1} and U_{w2}. The flow is driven by buoyancy forces and/or a common pressure gradient. The fluids present different densities ρ_i, viscosities μ_i, electrical and thermal conductivities, σ_i and k_i, and magnetic permeability μ_{ei}.

A constant magnetic field of strength B_0 is applied in a direction that forms an angle α with the y^* axis. An induced magnetic field of strength B^*_x is generated along the flow direction due to the fluid motion. The total magnetic field is

$$\bar{B}^* = \left(B_x^*(y^*) + B_0\sqrt{1-\lambda^2}\right)\bar{i} + B_0\lambda\bar{j},$$

where $\lambda = \cos\alpha$. As small magnetic Reynolds numbers were assumed, this induced field does not affect the flow field, i.e., $B^*_x << B_0$. The induced field is evaluated to demonstrate its connection with the flow field.

Hall and ion-slip effects are neglected herein. In addition to viscous and Joule heating, other effects can be incorporated into the model as source/sink terms (porous media, κ_{pi}; heat generation/absorption, q_i). All properties are constant except for the buoyancy term.

By defining the following dimensionless groups,

$$u_i = u_i^*/\bar{u}_1;\quad y = y^*/h_i;\quad \theta_i = (T_i - T_{w2})/(T_{w1} - T_{w2});$$
$$m = \mu_1/\mu_2;\quad k_c = k_1/k_2;\quad \kappa_c = \kappa_{p1}/\kappa_{p2};\quad h = h_2/h_1;$$
$$r = \rho_2/\rho_1;\quad b = \beta_2/\beta_1;\quad s = \sigma_2/\sigma_1;\quad m_e = \mu_{e2}/\mu_{e1};$$
$$\mathrm{Re} = \bar{u}_1 h_1/v_1;\ P = h_1^2/(\mu_1\bar{u}_1)\partial p^*/\partial x^*;\quad \mathrm{Pr} = \mu_1 c_{p1}/k_1;$$
$$Ec = \bar{u}_1^2/\left(c_p(T_{w1} - T_{w2})\right);\ Q_i = q_i\, h_i^2/k_i;\quad \varepsilon_i = h_i\Big/\sqrt{\kappa_{pi}};$$
$$Ha_i = B_0 h_i\sqrt{\sigma_i/\mu_i};\quad E_z = E_z^*/(B_0\bar{u}_1);\ B_{xi} = B_{xi}^*/B_0;$$
$$Gr = g\beta_1 h_1^3(T_{w1} - T_{w2})/v_1^2;\ Rm_1 = \bar{u}_1 h_1\sigma_1\mu_{e1}$$

the dimensionless governing equations for the laminar, fully-developed mixed MHD-convection are written for phase/region i (i = 1, 2) as:

$$\frac{d^2u_i}{dy^2} + h^2 m\left(rb\frac{Gr}{\mathrm{Re}}\theta_i\sin\phi - P\right) - Ha_i^2\lambda(E_z + u_i\lambda) - \varepsilon_i^2 u_i = 0 \qquad (1)$$

$$\frac{d^2\theta_i}{dy^2} + Q_i\theta_i + Ec\,\mathrm{Pr}\frac{k_c}{m}\left[\left(\frac{du_i}{dy}\right)^2 + Ha_i^2(E_z + u_i\lambda)^2 + \varepsilon_i^2 u_i^2\right] = 0 \qquad (2)$$

$$\frac{d^2B_{xi}}{dy^2} + shm_e\lambda Rm\frac{du_i}{dy} = 0 \qquad (3)$$

These equations are subject to the boundary and interface conditions as follows:

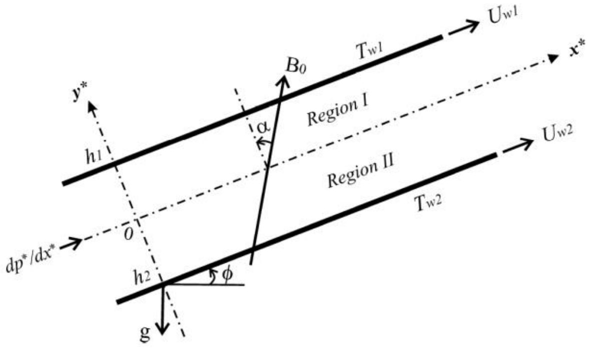

Figure 1. Sketch of the physical configuration studied.

$$\left.\begin{aligned} u_1(1) &= u_{w1}\\ \theta_1(1) &= \theta_w\\ B_{x1}(1) &= 0\end{aligned}\right\}, y = 1;\quad \left.\begin{aligned} u_2(-1) &= u_{w2}\\ \theta_2(-1) &= 0\\ B_{x2}(-1) &= 0\end{aligned}\right\}, y = -1 \qquad (4.a\text{-}f)$$

$$\left.\begin{aligned} u_1(0) = u_2(0);\quad & \left.\frac{du_1}{dy}\right|_{y=0} = \frac{1}{mh}\left.\frac{du_2}{dy}\right|_{y=0}\\ \theta_1(0) = \theta_2(0);\quad & \left.\frac{d\theta_1}{dy}\right|_{y=0} = \frac{1}{k_c h}\left.\frac{d\theta_2}{dy}\right|_{y=0}\\ B_{x1}(0) = B_{x2}(0);\quad & \left.\frac{dB_{x1}}{dy}\right|_{y=0} = \frac{1}{shm_e}\left.\frac{dB_{x2}}{dy}\right|_{y=0}\end{aligned}\right\}, y = 0 \qquad (4.g\text{-}l)$$

For Region I, coefficients b, m, r, h, k_c, k_p, s and m_e must be equal to unity (by using the aforementioned dimensionless parameters, the coefficients do not appear in the dimensionless equations for this region, see Eqs. (5-10). Re, Pr, Gr, Ec, Ha and Rm are the Reynolds, Prandtl, Grashof, Eckert, Hartmann and magnetic Reynolds numbers, respectively. The transversal y^* coordinate is made dimensionless through the respective region height according to the dimensional transversal position ($-h_2 \le y^* \le 0 \rightarrow y = y^*/h_2$, $0 \le y^* \le h_1 \rightarrow y = y^*/h_1$). E_z is a transversal imposed/generated electrical field. u_{w1} and u_{w2} are the dimensionless upper and lower wall velocities, respectively. θ_w is a dimensionless temperature parameter, which can be equal to zero or one depending on whether T_{w1} is equal to or different from T_{w2}.

The procedure used by Van Gorder *et al.* (2012) is adopted to map both fluid regions in a single domain [0,1]. The following relations are defined: $\hat{u}_2(y) = u_2(-y)$, $\hat{\theta}_2(y) = \theta_2(-y)$, $\hat{B}_{x2}(y) = B_{x2}(-y)$, which map Region II from domain [-1,0] to [0,1], and the resulting system is written in a single domain as follows:

$$-P + \frac{d^2u_1}{dy^2} + \frac{Gr}{\mathrm{Re}}\theta_1\sin\phi - Ha_1^2\lambda(E_z + u_1\lambda) - \varepsilon_1^2 u_1 = 0 \qquad (5)$$

$$\frac{d^2\theta_1}{dy^2} + Q_1\theta_1 + Ec\,\mathrm{Pr}\left[\left(\frac{du_1}{dy}\right)^2 + Ha_1^2(E_z + u_1\lambda)^2 + \varepsilon_1^2 u_1^2\right] = 0 \qquad (6)$$

$$\frac{d^2B_{x1}}{dy^2} + \lambda Rm_1\frac{du_1}{dy} = 0 \qquad (7)$$

$$\frac{d^2\hat{u}_2}{dy^2} + mh^2\left(-P + rb\frac{Gr}{\mathrm{Re}}\hat{\theta}_2\sin\phi\right) - Ha_2^2\lambda(E_z + \hat{u}_2\lambda) - \varepsilon_2^2 u_2 = 0 \qquad (8)$$

$$\frac{d^2\hat{\theta}_2}{dy^2} + Q_2\hat{\theta}_2 + Ec\,\mathrm{Pr}\frac{k_c}{m}\left[\left(\frac{d\hat{u}_2}{dy}\right)^2 + Ha_2^2(E_z + \hat{u}_2\lambda)^2 + \varepsilon_2^2\hat{u}_2^2\right] = 0 \qquad (9)$$

$$\frac{d^2\hat{B}_{x2}}{dy^2} - shm_e\lambda Rm_1\frac{d\hat{u}_2}{dy} = 0 \qquad (10)$$

where $\varepsilon_2^2 = \kappa_p h^2\varepsilon_1^2$ and $Ha_2^2 = smh^2Ha_1^2$.

Now, the governing equations must satisfy the "boundary" conditions (at y = 0 and y = 1, respectively) as follows:

$$\left.\begin{array}{c} u_1(0)=\hat{u}_2(0) \\ \left.\dfrac{du_1}{dy}\right|_{y=0}=-\dfrac{1}{mh}\left.\dfrac{d\hat{u}_2}{dy}\right|_{y=0} \\ \theta_1(0)=\hat{\theta}_2(0) \\ \left.\dfrac{d\theta_1}{dy}\right|_{y=0}=-\dfrac{1}{k_c h}\left.\dfrac{d\hat{\theta}_2}{dy}\right|_{y=0} \\ B_{x1}(0)=\hat{B}_{x2}(0) \\ \left.\dfrac{dB_{x1}}{dy}\right|_{y=0}=-\dfrac{1}{shm_e}\left.\dfrac{d\hat{B}_{x2}}{dy}\right|_{y=0} \end{array}\right\}; \quad \begin{array}{c} u_1(1)=u_{w1} \\ \theta_1(1)=\theta_w \\ \hat{u}_2(1)=u_{w2} \\ \hat{\theta}_2(1)=0 \\ B_{x1}(1)=0 \\ \hat{B}_{x2}(1)=0 \end{array} \qquad (11.\text{a-l})$$

B. Solution Methodology

Due to the presence of coupling and non-linear terms (buoyancy force, Joulean and viscous dissipations, and porosity of the medium), there is no exact analytical solution for this general model.

Herein the proposal is to solve the second-order ODE system as a coupled first-order ODE system using methods for the stiff two-point boundary-value problems instead of approximating analytical methods (Malashetty *et al.*, 2004; Umavathi *et al.*, 2010) or discretizing the system through direct numerical methods, such as the finite-difference, finite-volume or finite-element methods (Umavathi *et al.*, 2005; Rawat *et al.*, 2014). Herein, the well-documented BVPFD routine (IMSL, 2006) was applied. The BVPFD routine solves a parameterized system of ODE with boundary conditions at two points using a variable-order, variable-step-size finite difference method with deferred corrections (Pereyra, 1978).

Accordingly, from the relationship,

$$\begin{aligned} \gamma &= \{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\gamma_8,\gamma_9,\gamma_{10},\gamma_{11},\gamma_{12}\} \\ &= \left\{u_1,\frac{du_1}{dy},\theta_1,\frac{d\theta_1}{dy},\hat{u}_2,\frac{d\hat{u}_2}{dy},\hat{\theta}_2,\frac{d\hat{\theta}_2}{dy},B_{x1},\frac{dB_{x1}}{dy},\right. \\ &\qquad \left.\hat{B}_{x2},\frac{d\hat{B}_{x2}}{dy}\right\} \end{aligned} \qquad (12)$$

the original system of 6 second-order differential equations is transformed into a coupled system of 12 first order differential equations as:

$$\frac{d\gamma_j}{dy}=\gamma_{j+1}; \qquad j=1,3,4,7,9,11 \qquad (13.\text{a})$$

$$\frac{d\gamma_j}{dy}=mh^2\left[P-rb\frac{Gr}{\text{Re}}\gamma_{j+1}\sin\phi+sHa_1^2\lambda\left(E_z+\lambda\gamma_{j-1}\right)\right]+ \kappa_p h^2\varepsilon_1^2\gamma_{j-1}; \qquad j=2,6 \qquad (13.\text{b})$$

$$\frac{d\gamma_j}{dy}=-Ec\,\text{Pr}\frac{k_c}{m}\left[\gamma_{j-2}^2+smh^2Ha_1^2\left(E_z+\lambda\gamma_{j-3}\right)^2+ \kappa_p h^2\varepsilon_1^2\gamma_{j-3}^2\right]-Q_{1,2}\theta_1\gamma_{j-1}; \qquad j=4,8 \qquad (13.\text{c})$$

$$\frac{d\gamma_{10}}{dy}=-\lambda Rm_1\gamma_2; \qquad \frac{d\gamma_{12}}{dy}=shm_e\lambda Rm_1\gamma_6 \qquad (13.\text{d,e})$$

As already explained, for j = 2 and 4, coefficients b, m, r, h, k_c, κ_p, s and me must be equal to one in Eqs. (13.b, c). This first-order coupled system is now subject to the following "left" (y=0) and "right" (y=1) boundary conditions, respectively, as follows:

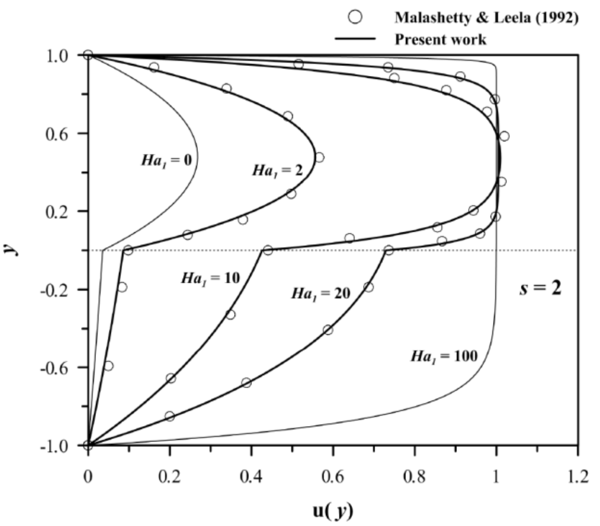

Figure 2. Velocity profiles versus Hartmann numbers.

$$\left.\begin{array}{c} \gamma_1(0)-\gamma_5(0)=0 \\ \gamma_2(0)+\dfrac{1}{mh}\gamma_6(0)=0 \\ \gamma_3(0)-\gamma_7(0)=0 \\ \gamma_4(0)+\dfrac{1}{k_c h}\gamma_8(0)=0 \\ \gamma_9(0)-\gamma_{11}(0)=0 \\ \gamma_{10}(0)+\dfrac{1}{shm_e}\gamma_{12}(0)=0 \end{array}\right\}; \quad \begin{array}{c} \gamma_1(1)-u_{w1}=0 \\ \gamma_3(1)-\theta_w=0 \\ \gamma_5(1)-u_{w2}=0 \\ \gamma_7(1)=0 \\ \gamma_9(1)=0 \\ \gamma_{10}(1)=0 \end{array} \qquad (14.\text{a-f})$$

C. Results and Discussion

The resulting first-order system was implemented in a Fortran 90 language code. Potentials were solved using the BVPFD routine for a relative error target of 10^{-10}; i.e., all potentials and their first-order derivatives were exactly evaluated to the 10^{th} precision digit. Numerical results are presented, illustrating the behavior of the flow, magnetic and heat transfer fields for several values of the 24 governing dimensionless parameters (P, Gr, Re, E_z, ϕ, Ha_1, ε_1, Ec, Pr, m, h, r, b, s, k_c, κ_p, Q_1, Q_2, u_{w1}, u_{w2}, θ_w, λ, m_e and R_{m1}) for different physical situations. If Ha_1 and/or ε_1 are zero, the user can enter any value for Ha_2 and/or ε_2 if region II is an electrical conducting and/or porous medium.

The velocity profiles for the fully-developed forced (P=-2,Gr=0) MHD convection in a non-porous (ε_1 =0, κ_p =0) horizontal channel (ϕ = 0) with stationary plates (u_{w1}=u_{w2}= 0) at the same temperature (θ_w = 0) are validated against the analytical results of Malashetty and Leela (1992) in Fig. 2 for different values of Ha_1 and for Re=1; E_z=-1; Ec = 1; Pr = 1; m = 0.333; h=0.1; r=1; b =1; s =2; k_c =1; Q_1= Q_2 = 0; λ =1; m_e =1; R_{m1} = 0.

For this open-circuit case (E_z = -1, imposed), Fig. 2 shows that the Lorentz force accelerates the flow in the channel (E_z must be greater than λu in magnitude), in contrast to $E_z \geq 0$. As λ= 1 herein, the external magnetic field is perpendicularly applied to the flow direction. A stronger Hartmann number corresponds to flatter velocity profiles in both regions (compare the limiting situations for Ha = 0 and Ha = 100: as Ha increases, the electrical conductivity ratio practically loses influence on region profiles). It must be noted that the dissimilar behavior between profiles for regions I and II is mainly due to the viscosity and height ratios (m=0.333; h=0.1), irrespective the value of the electrical conductivity ratio. Figure 3 illustrates the velocity results of the mixed MHD convection (P= -5, Gr =5) in an inclined nonporous channel (ε_1 =0, κ_p =0) with stationary plates at different temperatures ($u_{w1} = u_{w2} = 0$, θ_w= 1) for different channel inclinations. The results of Malashetty *et al.* (2001) were utilized for comparison purposes, for Re=5; Ha_1 = 2; Ec = 10^{-5}; Pr = 1; m = 1.0; h = 1; r = 1.5; b=1; s=2; k_c =1; Q_1= Q_2 =0; m_e =1 and R_{m1}=0. Despite the channel inclination, the external magnetic field is always applied in a direction normal to the flow (λ =1).

The results are consistent with the approximate results of Malashetty *et al*. (2001) for all studied inclinations. Figure 3 shows that channel inclination (and therefore natural convection effects) has a lower influence on flow profile in the region where the retarding Lorentz force is higher (region II, where the electrical conductivity is higher and E_z = 0).

Regarding the effect of porosity and height ratios on flow behavior, Malashetty *et al.* (2004) studied the non-MHD (Ha_1= Ha_2=0) two-fluid mixed convection (P=-5; Gr=5) in an inclined channel with a porous top layer ($\varepsilon_1 \neq 0$) and a non-porous bottom layer ($\varepsilon_2 \neq 0$) using the regular perturbation method. As in the previous case, the magnetic field is applied normally to the flow (λ=1). Channel plates are stationary and at different temperatures (u_{w1}=u_{w2}=0, θ_w=1).

Comparisons with the results of Malashetty *et al.* (2004) are illustrated in Fig. 4 for Re=5; E_z=0; ϕ=30º; Ec = 10^{-5}; Pr = 1; m= 1; r = 1.5; b = 1; k_c = 1; Q_1= Q_2 =0; m_e =1, R_{m1}=0 and different values of h and ε_1.

The results obtained herein are consistent with those of Malashetty *et al.* (2004), despite slight differences in the center of region II for the greater channel height ratio. This behavior could be due to mistyped values of some dimensionless parameters. Herein results were obtained for m = 1, whereas Malashetty *et al.* (2004) indicate m=0.5.

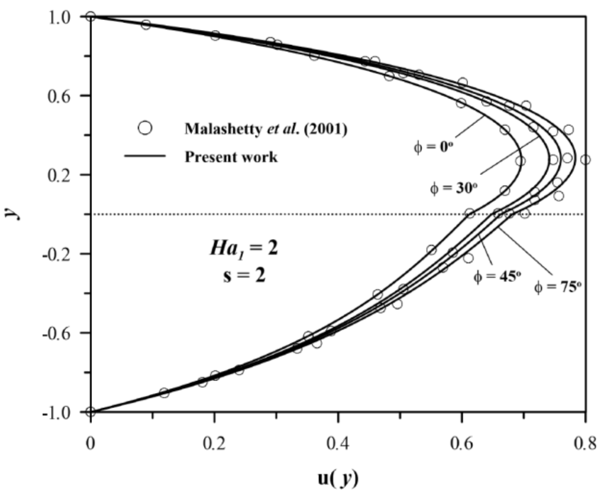

Figure 3. Velocity profiles versus channel inclinations.

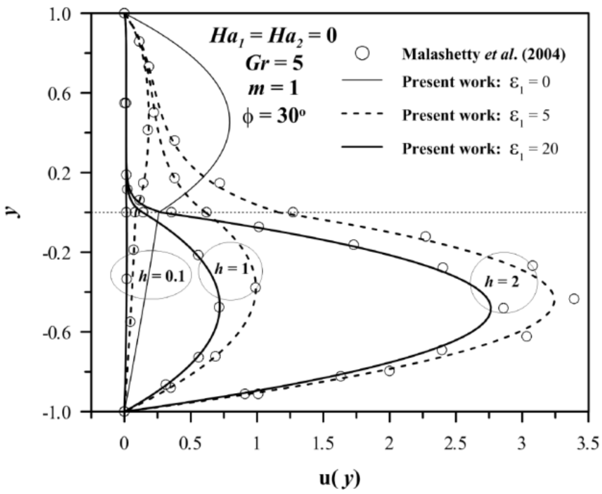

Figure 4. Velocity profiles for different top layer porosities and layer height ratios for non-MHD flow.

The terms relative to porosity in the governing equations closely resemble the Lorentz force and Joule dissipation effects. The porosity parameter can be interpreted as an equivalent Hartmann number, i.e., $\varepsilon_1 \equiv \lambda\ Ha_1$ (only for the particular case of the fully developed MHD flow with a small magnetic Reynolds number and E_z=0). The effect of increasing the porosity parameter, ε_1, is identical to the effect of increasing the Hartmann number: the flow field is retarded (the results of Fig. 4 can also be obtained by substituting ε_1 by $\lambda\ Ha_1$ and κ_p by s values). Conversely, an increase in the height ratio implies in a decrease in the influence of the porosity of region I, and therefore a retarding effect is produced on the flow profile of region II.

Analysis now considers the mixed convection (P=-5; Gr=5) in a vertical (ϕ=90º) non-porous channel (ε_1=ε_2=0) of two immiscible and electric conducting fluids (Ha_1=4; s=2). This situation was investigated by Umavathi *et al.* (2005), by using the regular perturbation method for small values of the product $Ec.Pr$, and the finite-difference method for the higher values. In addition to Joule and viscous dissipation, volumetric heat source/sink can occur.

The velocity profiles are compared in Fig. 5 as functions of the electrical parameter and Eckert number for the heat generation case (Q_1=Q_2=1) and Ha_1=4; Re=5; Pr=1; m=1; h=1; r=1; b=1; k_c=1; m_e=1 and R_{m1}=0. The two vertical plates are at different temperatures and are stationary (u_{w1}=u_{w2}=0, θ_w=1).

The results are consistent with the finite-difference findings of Umavathi *et al*. (2005) for all values of the analyzed parameters. It must be highlighted that if E_z=1 and Ec>0.7, the numerical solution strongly depends (by numerical or physical reasons) on the initial guess, i.e., the ODE system becomes extremely stiff. The results for Ec>0.7 are only obtained correctly with the use of an adaptive procedure, where results for smaller Eckert numbers were used as the initial guess for the higher numbers.

Apart from this numerical issue, Fig. 5 clearly illustrates the effect of the electrical parameter, E_z, which in conjunction with Hartmann number, governs the magnetohydrodynamic characteristics of the channel (i.e., MHD generator or pump).

Umavathi *et al.* (2010) applied the perturbation method to study mixed convection in an inclined channel with a moving top plate. Only the fluid in the bottom side of the channel was electrically conducting.

The comparisons are illustrated in Fig. 6, which shows the effect of Grashof number on the velocity field. The following set was used: P=-5; Re=1; E_z=0; ϕ=30º; ε_1=0; Ec=0.01; Pr=0.7; m=1; h=1; r=1; b=1; k_c=1; κ_p=0 (ε_2=0); Q_1=Q_2=0; u_{w1}=1; u_{w2}=0; θ_w=1; λ=1; m_e=1 and R_{m1} = 0. As only the fluid in Region II is electrically conducting, Ha_1=0; Ha_2=2 was used in the computations.

The expected behavior of the velocity field with Grashof number is naturally predicted by the present approach, whose results are consistent with those obtained by Umavathi *et al*. (2010). The limiting cases of MHD-Couette (P=0; Gr=0) and forced-convection (P=-5, Gr=0) flows are also illustrated. As the fluid in region II is electrical conducting and E_z= 0, flow is retarded in this region.

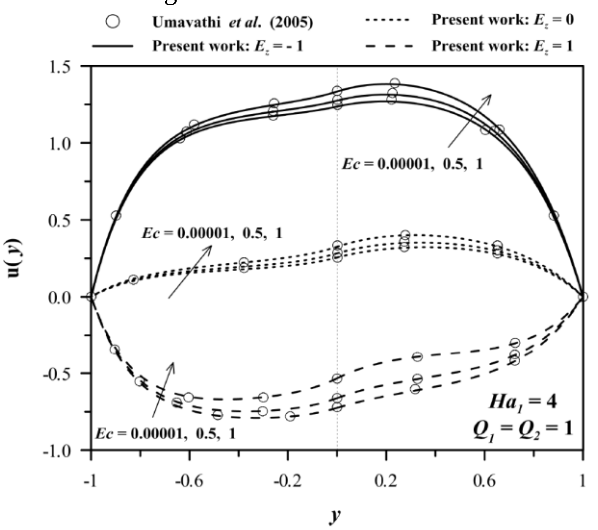

Figure 5. Velocity profiles for different pairs of Eckert number and electrical parameter. Heat generation case.

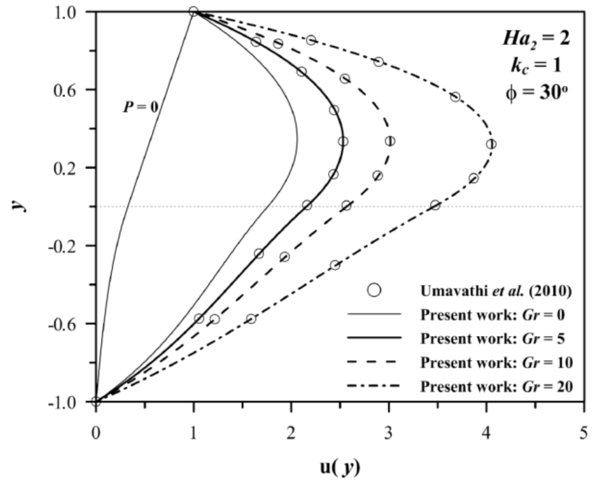

Figure 6. Velocity profiles versus Grashof numbers.

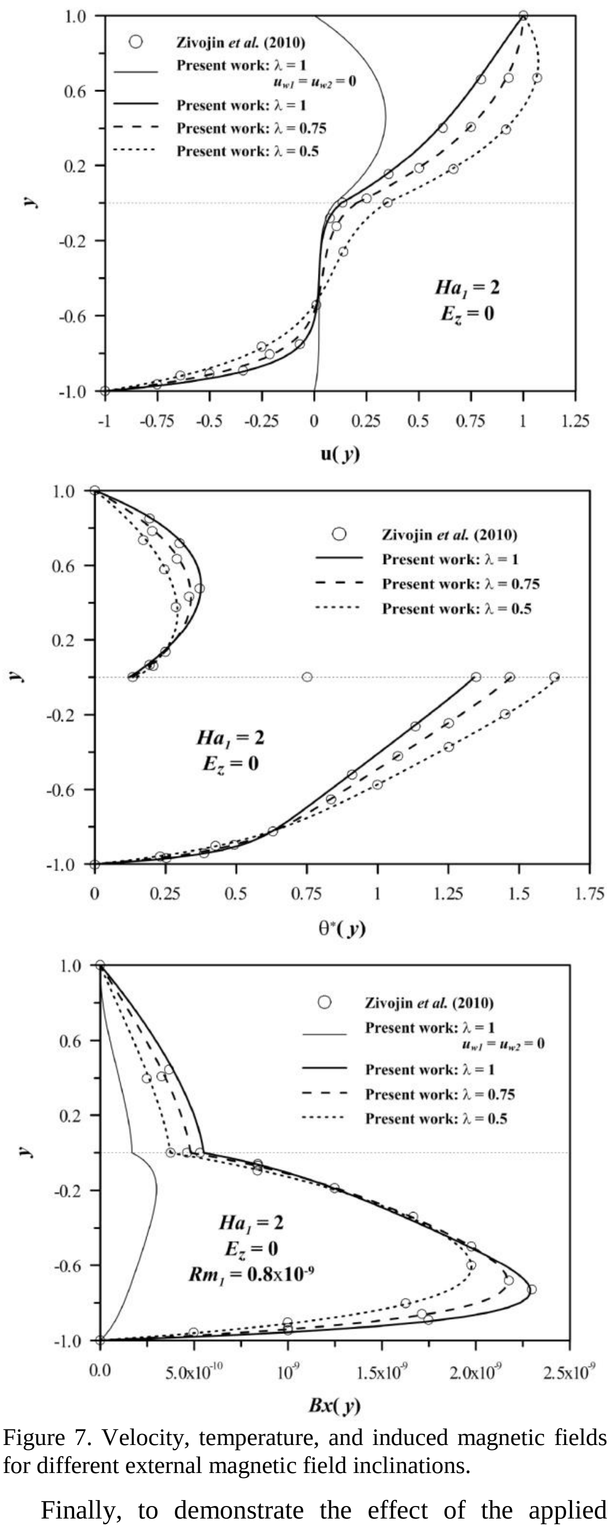

Figure 7. Velocity, temperature, and induced magnetic fields for different external magnetic field inclinations.

Finally, to demonstrate the effect of the applied magnetic field inclination on the flow, temperature and induced magnetic fields, Fig. 7 compares these potentials with the results of Zivojin *et al*. (2010).

Zivojin *et al*. (2010) analytically studied the fully developed forced convection (P=-3.5; G=0) in a MHD generator channel where two-immiscible electrically conducting fluids (Ha_1=2; =37.9) flowed in a horizontal channel (ϕ=0). Both channel plates were at the same temperature (θ_w=1) and moved with constant velocity in opposite directions (u_{w1}=-u_{w2}=1). Results were obtained by considering Re = 1; E_z=0; ε_1=0; Ec=1; Pr=1; m=0.66; h=1; r=1; b=1; k_c=0.06; κ_p=0 (ε_2=0); Q_1=Q_2=0; m_e=1 and R_{m1}=0.8×10^{-9}.

The findings herein obtained were consistent with the analytical solutions for all analyzed fields. The effect of the opposite top- and bottom-plate velocities on the flow behavior can be observed: the point where the velocity is zero is displaced from the channel center. This occurred because the viscosity ratio, m, is different from one regardless of the inclination of the external magnetic field. Another noteworthy effect is the artificial jump in temperature at the interface (y=0), which results from the selection of the dimensionless groups herein: $\theta_1^*=(\theta_1-1)/(Ec\ \mathrm{Pr})$, $\theta_2^*=\theta_2\ m/(k_c\ Ec\ \mathrm{Pr})$. The results for $u_{w1}=u_{w2}=0$ illustrate the strong dependence of the induced magnetic field on the flow field.

III. CONCLUSIONS

The results obtained herein were validated against analytical solutions and compared with other approximate numerical methods. The comparison clearly demonstrated the simplicity and efficiency of the numerical route presented herein to analyze a broad spectrum of problems that are governed by the two-point boundary value ordinary differential equations, in particular, those related to the fully developed magnetohydrodynamic flow of two-immiscible fluids. The method is mainly applicable to problems where no analytical solution is possible or is difficult to obtain.

Hybrid methods such as the Generalized Integral Transform Technique (Lima *et al*., 2007; Lima and Rêgo, 2013) use intensive numerical routines for the initial and boundary value problems to solve the resulting transformed ODE systems. Thus, the work herein presented serves as a physical and numerical starting point for the application of such techniques to develop two-phase heat and fluid MHD-flow studies in channels.

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REFERENCES

Davidson, P.A., *An Introduction to Magnetohydrodynamics*, Cambridge University Press, Cambridge, UK (2001).

IMSL *Fortran Numerical Libraries*, Version 6.0, Visual Numerics Inc. (2006).

Lima, J.A., J.N.N. Quaresma and E.N. Macêdo, "Integral transform analysis of MHD flow and heat transfer in parallel plates channels," *Int. Comm. Heat Mass Transfer*, **34**, 420-431 (2007).

Lima, J.A. and M.G.O. Rêgo, "On the integral transform solution of low-magnetic MHD flow and heat transfer in the entrance region of a channel," *Int. J. Non-Linear Mechanics*, **50**, 25-39 (2013).

Malashetty, M.S. and V. Leela, "Magnetohydrodynamic heat transfer in two phase flow," *Int. J. Engineering Science*, **30**, 371-377 (1992).

Malashetty, M.S. and J.C. Umavathi, "Two-phase magnetohydrodynamic flow and heat transfer in an inclined channel," *Int. J. Multiphase Flow*, **22**, 545-560 (1997).

Malashetty, M.S., J.C. Umavathi and J.P. Kumar, "Convective magnetohydrodynamic two fluid flow and heat transfer in an inclined channel," *Heat and Mass Transfer*, **37**, 259-264 (2001).

Malashetty, M.S., J.C. Umavathi and J.P. Kumar, "Two fluid flow and heat transfer in an inclined channel containing porous and fluid layer," *Heat and Mass Transfer*, **40**, 871-876 (2004).

Malashetty, M.S., J.C. Umavathi and J.P. Kumar, "Magnetoconvection of two-immiscible fluids in vertical enclosure," *Heat and Mass Transfer*, **42**, 977-993 (2006).

Murty, P.S.R. and G.B. Prakash, "MHD Two-Fluid and Heat Transfer between Two Inclined Parallel Plates in a Rotating System," *International Scholarly Research Notices*, Article ID 256898 (2014).

Nikodijevic, D., D. Milenkovic and Z. Stamenkovic, "MHD Couette two-fluid flow and heat transfer in presence of uniform inclined magnetic field," *Heat and Mass Transfer*, **47**, 1525-1535 (2011).

Pereyra, V., "PASVA3: An adaptive finite-difference FORTRAN program for first order nonlinear boundary value problems," *Lectures Notes in Computer Science*, Springer-Verlag, **76**, 67-88 (1978).

Rawat, S., R. Bhargava, S. Kapoor, O.A. Bég, T.A. Bég, and R. Bansal, "Numerical Modelling of Two-Phase Hydromagnetic Flow and Heat Transfer in a Particle-Suspension through a non-Darcian Porous Channel," *Journal of Applied Fluid Mechanics*, **7**, 249-261 (2014).

Umavathi, J.C., A.J. Chamka, M.H. Manjula, and A. Al-Mudhai, "Magneto-convection of a two-fluid flow through a vertical channel," *Heat and Technology*, **23**, 151-163 (2005).

Umavathi, J.C., I-C. Liu and J.P. Kumar, "Magnetohydrodynamic Poiseuille-Couette flow and heat transfer in an inclined channel," *Journal of Mechanics*, **26**, 525-532 (2010).

Zivojin, S.M., D.D. Nikodijevic, B.D. Blagojevic and S.R. Savic, "MHD flow and heat transfer of two immiscible fluids between moving plates," *Trans. Can. Society for Mechanical Engineering*, **23**, 351-372 (2010).

Van Gorder, R., K.V. Prasad and K. Vajravelu, "Convective heat transfer in the vertical channel flow of a clear fluid adjacent to a nanofluid layer: a two-fluid model," *Heat & Mass Transfer*, **48**, 1247-1255 (2012).

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