ENTROPY GENERATION IN MHD FLOW THROUGH A DEFORMABLE POROUS LAYER WITH CONSTANT INJECTION/ SUCTION AT THE BOUNDARY

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***Abstract*— The present paper examines the entropy generation on MHD flow of viscous fluid over a deformable vertical porous layer with constant injection/ suction velocity at the boundary walls of the layer. The combined phenomenon of the solid deformation and fluid movement in the porous medium are taken into consideration. The influence of relevant non-dimensional parameters on the fluid velocity, solid displacement, temperature and concentration profiles are discussed. Also, the impact of Brinkman number, volume fraction parameter and drag parameter on entropy generation and Bejan number are discussed.**

***Keywords*— MHD, deformable porous layer, entropy generation, Brinkman number, Bejan number.**

I. INTRODUCTION

The study of viscous fluid flow through deformable porous layer has enormous applications in the fields of geothermal regions and engineering. Also, it has many important applications in the problems of tissue mechanics for articular cartilage. The works studied through deformable porous media is found to be very limited. The problem of combined phenomena of solid deformation and fluid flow through porous media was initiated by Terzahi (1925). Further, Biot (1955, 1956) developed this theory. Using this theory, arterial wall permeability and articular cartilage have been modelled by Kenyon (1979), Jayaraman (1983), Jain and Jayaraman (1987) and Holmes and Mow (1990). Barry *et al.* (1991) examined viscous fluid flow through a thin deformable porous layer. Klubertanz *et al.* (2003) studied multiphase flow of Newtonian fluid in deformable porous media. Ranganatha and Siddagamma (2004) studied Newtonian fluid flow in a channel flow of deformable porous walls. Sreenadh *et al.* (2015, 2016) studied the Jeffrey fluid flow over a deformable porous layer. Krishna Murthy (2019) analyzed heat and mass transfer of MHD Casson liquid over a deformable porous medium considering slip effects. Recently, Rajashekar *et al.* (2020), Vaidya *et al.* (2020a, 2020b, 2021) have presented peristaltic flow in channel through undeformable porous media with different physical properties.

Entropy generation of a fluid flow system gives an insight of the power consumption due to thermodynamic losses. Optimization of power for fluid motion is achieves by minimizing the entropy generation. Woods (1975) studied the thermodynamic properties of fluid. Bejan (1980) explained the optimization of entropy generation for fluid flow system. Egunjobi and Makinde (2013) studied the entropy generation for MHD flow of Newtonian fluid in a channel. Sreenadh *et al.* (2018) studied the MHD Newtonian fluid flow through deformable vertical porous layer.

In this study, MHD flow through deformable porous layer in presence of first-order chemical reaction with constant injection/ suction at the permeable boundary has been investigated. The expression representing solutions for fluid velocity, solid displacement, temperature and concentration distributions are obtained.

II. MATHEMATICAL FORMULATION

Consider a fully developed flow of steady, incompressible viscous fluid through a deformable, vertical porous layer with constant injection/ suction at the porous walls in presence of first order chemical reaction. The porous material is designed as a continuous, homogeneous and isotropic mixture of fluid and solid phases where each point in the binary mixture is occupied concurrently by fluid and solid. The $\bar{x}$- axis is considered along one of walls and $\bar{y}$-axis at right angles to it as shown in Fig.1.

The governing equations of the flow, following Barry *et al.* (1991) and Sreenadh *et al.* (2016) become

$$\mu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + kv - (1-\phi)\frac{\partial \bar{P}}{\partial \bar{x}} = 0, \tag{1}$$

$$\rho^f V \frac{\partial \bar{v}}{\partial \bar{y}} + 2\mu_a \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + g\rho^f \beta_T(\bar{T} - \bar{T}_0) + g\rho^f \beta_C(\bar{C} - \bar{C}_0) - \phi \frac{\partial \bar{P}}{\partial \bar{x}} - K\bar{v} - \sigma B_0^2 \bar{v} = 0, \tag{2}$$

$$K_0 \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + V\frac{\partial \bar{T}}{\partial \bar{y}} + 2\mu_a \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + K\bar{v}^2 = 0, \tag{3}$$

$$V\frac{\partial \bar{C}}{\partial \bar{y}} + D\frac{\partial^2 \bar{C}}{\partial \bar{y}^2} - \bar{R}(\bar{C} - \bar{C}_0) = 0. \tag{4}$$

The boundary conditions are

$$\bar{u} = 0, \bar{v} = 0, \bar{T} = T_w, \bar{C} = C_w \text{ at } \bar{y} = a;$$

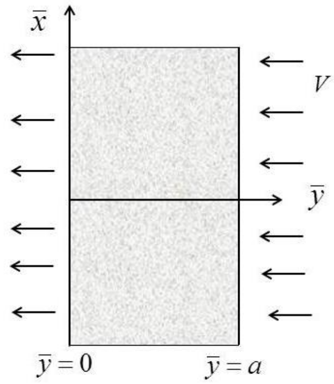

Figure1: Physical model

$$\frac{\partial \bar{u}}{\partial \bar{y}}=0, \frac{\partial \bar{v}}{\partial \bar{y}}=0, \frac{\partial \bar{T}}{\partial \bar{y}}=0, \frac{\partial \bar{C}}{\partial \bar{y}}=0 \text{ at } \bar{y}=0, \quad (5)$$

where $\bar{u}$ is the solid displacement and $\bar{v}$ is the fluid velocity. All the variables are explained in the Nomenclature.

Following non-dimensional quantities are introduced:

$$y=\frac{\bar{y}}{a}, v=\frac{\bar{v}}{V}, u=\frac{\bar{u}\mu}{\mu_f V}, x=\frac{\bar{x}}{a}, T=\frac{\bar{T}-\bar{T}_0}{T_w-\bar{T}_0}, C=\frac{\bar{C}-\bar{C}_0}{C_w-\bar{C}_0},$$
$$P=\frac{a\bar{P}}{\mu_f V}, \mathrm{Re}=\frac{Va}{\nu_f}, \eta=\frac{\mu_f}{2\mu_a}, \delta=\frac{Ka^2}{\mu_f},$$
$$Gr=\frac{g\beta_T\rho^f a^2(T_w-\bar{T}_0)}{\mu_f V}, Gc=\frac{g\beta_C\rho^f a^2(C_w-\bar{C}_0)}{\mu_f V}, R=\frac{\bar{R}a^2}{D},$$
$$Pe=\frac{Va}{K_0}, M=\sqrt{\frac{a^2\sigma B_0^2}{\mu_f\rho^f}}, Br=\frac{2\mu_a V^2}{(T_w-\bar{T}_0)K_0}. \quad (6)$$

Using (6) in Eqs. (1)-(4), we get

$$\frac{d^2u}{dy^2}-(1-\phi)P+\delta v=0, \quad (7)$$
$$\frac{d^2v}{dy^2}+\mathrm{Re}\,\eta\frac{dv}{dy}-\phi P\eta+\eta GrT+\eta GcC - (\delta+M)\eta v=0 \quad (8)$$
$$\frac{d^2T}{dy^2}+Pe\frac{dT}{dy}+Br\left(\frac{dv}{dy}\right)^2+Brv^2=0, \quad (9)$$
$$\frac{d^2C}{dy^2}+\mathrm{Re}Sc\frac{dC}{dy}+RC=0, \quad (10)$$

The boundary conditions are

$$u=0, v=0, T=1, C=1 \text{ at } y=1;$$
$$\frac{du}{dy}=0, \frac{dv}{dy}=0, \frac{dt}{dy}=0, \frac{dC}{dy}=0 \text{ at y= } 0. \quad (11)$$

Solving (10), we get

$$C=A_1\left\{\exp(-m_1y)-\frac{m_1}{m_2}\exp(-m_2y)\right\}, \quad (12)$$

where

$$m_1=\frac{\mathrm{Re}Sc+\sqrt{\mathrm{Re}^2Sc^2+4R}}{2}, m_2=\frac{\mathrm{Re}Sc-\sqrt{\mathrm{Re}^2Sc^2+4R}}{2}.$$

The Eqs. (7)-(9), cannot be solved in closed form, so, to solve these non-linear coupled equations for small Br ($Br \ll 1$), we take

$$(u,v,T)=(u_0,v_0,T_0)+Br(u_0,v_0,T_0)+o(Br^2) \quad (13)$$

Using (13) in (7)-(9), we obtain the following system of equations with different order.

System of order Br^0:

$$\frac{d^2u_0}{dy^2}-(1-\phi)P+\delta v_0=0, \quad (14)$$
$$\frac{d^2v_0}{dy^2}+\mathrm{Re}\,\eta\frac{dv_0}{dy}-(\delta+M)\eta v_0=\phi P\eta -\eta GrT_0-\eta GcC=0 \quad (15)$$
$$\frac{d^2T_0}{dy^2}+Pe\frac{dT_0}{dy}=0. \quad (16)$$

The boundary conditions are

$$u_0=0, v_0=0, T_0=1 \text{ at } y=1;$$
$$\frac{du_0}{dy}=0, \frac{dv_0}{dy}=0, \frac{dT_0}{dy}=0 \text{ at y= } 0. \quad (17)$$

System of order Br^1:

$$\frac{d^2u_1}{dy^2}+\delta v_1=0, \quad (18)$$
$$\frac{d^2v_1}{dy^2}+\mathrm{Re}\,\eta\frac{dv_1}{dy}+\eta GrT_1-(\delta+M)\eta v_1=0 \quad (19)$$
$$\frac{d^2T_1}{dy^2}+Pe\frac{dT_1}{dy}=-\left(\frac{dv_0}{dy}\right)^2-v_0^2. \quad (20)$$

The boundary conditions are

$$u_1=0, v_1=0, T_1=0 \text{ at } y=1;$$
$$\frac{du_1}{dy}=0, \frac{dv_1}{dy}=0, \frac{dT_1}{dy}=0 \text{ at y= } 0. \quad (21)$$

Solving the Eqs. (14)-(16) and (18)-(20) with boundary conditions (17) and (21), we get

$$u_0=A_9\exp(-m_1y)+A_{10}\exp(-m_2y)+ A_{11}\exp(-m_3y)+A_{12}\exp(-m_4y)+ A_{13}y^2+A_7y+A_8, \quad (22)$$
$$v_0=A_3\exp(-m_1y)+A_4\exp(-m_2y)+ A_5\exp(-m_3y)+A_6\exp(-m_4y)+A_2, \quad (23)$$
$$T_0=1, \quad (24)$$
$$u_1=A_{77}y+A_{78}y^2+A_{79}y^3+A_{80}\exp(-Pey)+ A_{81}\exp(-2m_1y)+A_{82}\exp(-2m_2y)+ A_{83}\exp(-2m_3y)+A_{84}\exp(-2m_4y)+ A_{85}\exp(-m_1y-m_2y)+A_{86}\exp(-m_3y-m_1y)+ A_{87}\exp(-m_4y-m_1y)+A_{88}\exp(-m_3y-m_2y)+ A_{89}\exp(-m_2y-m_4y)+A_{90}\exp(-m_3y-m_4y)+ A_{91}\exp(-m_1y)+A_{92}\exp(-m_2y)+ A_{93}\exp(-m_3y)+A_{94}\exp(-m_4y)+A_{95}, \quad (25)$$
$$v_1=A_{44}+A_{45}\exp(-Pey)+A_{46}\exp(-2m_1y)+ A_{47}\exp(-2m_2y)+A_{48}\exp(-2m_3y)+ A_{49}\exp(-2m_4\,y)+A_{50}\exp(-m_3y-m_4y)+ A_{51}\exp(-m_3y-m_1y)+A_{52}\exp(-m_3y-m_2y)+ A_{53}\exp(-m_4y-m_1y)+A_{54}\exp(-m_4y-m_2y)+ A_{55}\exp(-m_1y-m_2y)+A_{56}\exp(-m_1y)+ A_{57}\exp(-m_2y)+A_{58}\exp(-m_3y)+ A_{59}\exp(-m_4y)+A_{60}y, \quad (26)$$
$$T_1=-(A_2^2/Pe)+A_{28}\exp(-2m_3y)+ A_{29}\exp(-2m_4y)+A_{30}\exp(-2m_1y)+ A_{31}\exp(-2m_2y)+A_{32}\exp(-m_3y-m_4y)+ A_{33}\exp(-m_3y-m_1y)+A_{34}\exp(-m_3y-m_2y)+ A_{35}\exp(-m_4y-m_1y)+A_{36}\exp(-m_4y-m_2y)+ A_{37}\exp(-m_1y-m_2y)+A_{38}\exp(-m_1y)+ A_{39}\exp(-m_2y)+A_{40}\exp(-m_3y)+ A_{41}\exp(-m_4y-m_1y)+A_{42}+A_{43}\exp(-Pey)$$
$$A_{41}\exp(-m_4y-m_1y)+A_{42}+A_{43}\exp(-Pey) \quad (27)$$

where

$$m_3=\frac{\eta\mathrm{Re}+\sqrt{\eta^2\mathrm{Re}^2+4(\delta+M)\eta}}{2}, m_4=\frac{\eta\mathrm{Re}-\sqrt{\eta^2\mathrm{Re}^2+4(\delta+M)\eta}}{2}.$$

and the constants A_i $(i=1,\dots,95)$ are presented in the Appendix..

The expression for displacement, velocity and temperature are given by

$$u=u_0+Bru_1, v=v_0+Brv_1, T=T_0+BrT_1.$$

III. ENTROPY GENERATION

Following Wood (1975), the local volumetric rate of entropy generation for flow of viscous incompressible fluid through a deformable porous layer is

$$E_G=\frac{K_0}{\bar{T}_0^2}\left(\frac{d\bar{T}}{d\bar{y}}\right)+\frac{2\mu_a}{\bar{T}_0}\left(\frac{d\bar{v}}{d\bar{y}}\right)^2+\frac{K}{\bar{T}_0}v^2. \quad (28)$$

The non-dimensional form of entropy generation number N_s is given as

$$N_s=\frac{\bar{T}_0^2a^2E_G}{\kappa(T_w-\bar{T}_0)^2}=\left(\frac{dT}{dy}\right)^2+\frac{Br}{\Omega}\left[\left(\frac{dv}{dy}\right)^2+v^2\right] = N_1+N_2$$

where $N_1=\left(\frac{dT}{dy}\right)^2$ is entropy generation due to heat transfer, $N_2=\frac{Br}{\Omega}\left[\left(\frac{dv}{dy}\right)^2+v^2\right]$ is entropy generation due to fluid friction and $\Omega=\frac{T_w-\bar{T}_0}{\bar{T}_0}$ is the non- dimensional temperature difference.

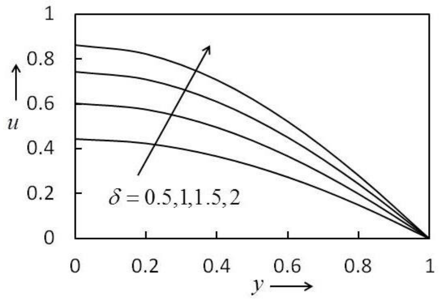

Figure 2: Displacement profile for different δ.

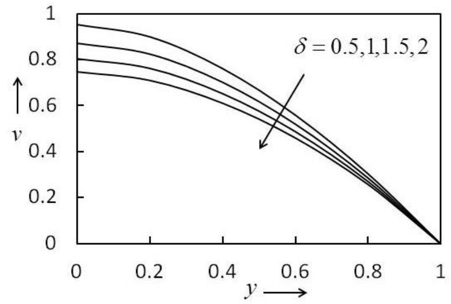

Figure 3: Velocity profile for different δ.

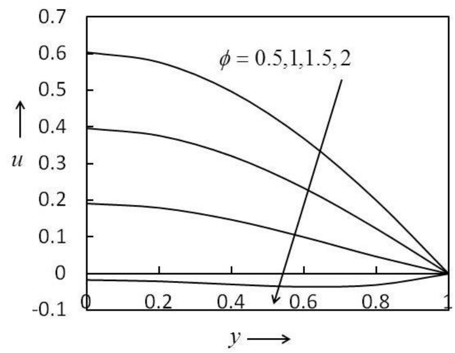

Figure 4: Displacement profile for different ϕ.

The Bejan number is defined as $Be = \frac{N_1}{N_2} = \frac{1}{1+\Phi}$, where $\Phi = \frac{N_2}{N_1}$ is irreversibility ratio.

IV. RESULTS AND DISCUSSION

Here, numerical calculations are carried out with the following values of the physical parameters at $Sc = 0.62$, $Pe = 0.5$, $\omega = 0.5$, $\mathrm{Re} = 2$, $\eta = 1$, $R = 1$, $Br = 0.1$, $M = 0.5$, $\delta = 1$, $P = -1$, $Gc = 2$, Gr=2.

Figures 2 and 3 represent the influence of drag parameter δ on solid displacement u and fluid velocity v. From Fig. 2 it is seen that u increase when δ increase, but Fig.3 indicates that v decrease due to the influence of δ. Physically, it can be interpreted that as there is less solid to impede during the growth of porosity of the medium and thus solid displacement decreases due to less drag on solid components. Figure 4 illustrates that due to increase in ϕ reduces the solid displacement whereas Fig. 5 depicts that the fluid velocity enhances when ϕ increase. Figures 6 and 7 shows that both solid displacement and fluid velocity reduces when pressure gradient increases. This can be physically explained considering that the increase of the pressure gradient makes the system thicker

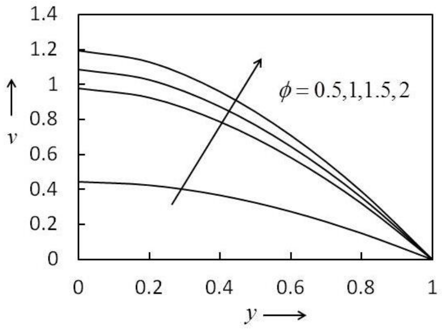

Figure 5: Velocity profile for different ϕ.

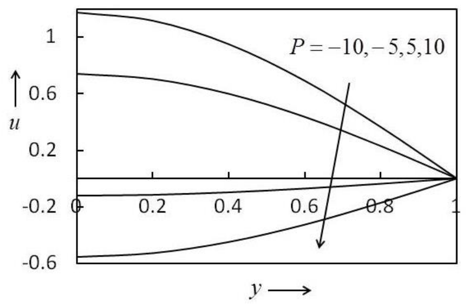

Figure 6: Displacement profile for different P.

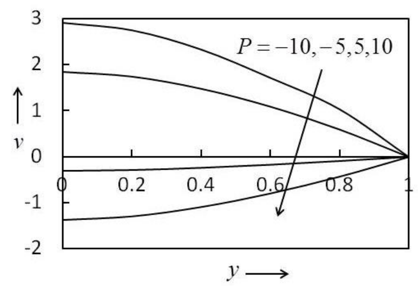

Figure 7: Velocity profile for different P.

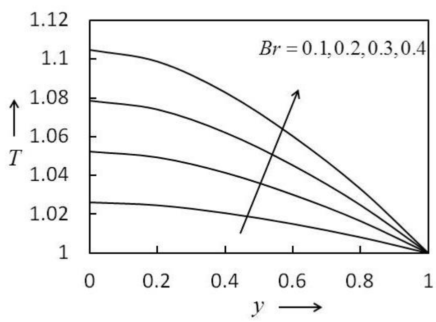

Figure 8: Temperature profile for different Br.

and consequently decreases the speed of solid displacement and fluid velocity.

Figure 8 shows that the temperature increases due to increase in Br. This is due to the fact that higher the value of Br, lower the conduction of heat produced by viscous dissipation and hence rises the temperature.

Figure 9 shows that concentration of species decrease when chemical reaction parameter increases.

Figure 10 shows that entropy generation number N_s increase when Br increases.

Figure 11 shows that Bejan number Be decrease when Br increases.

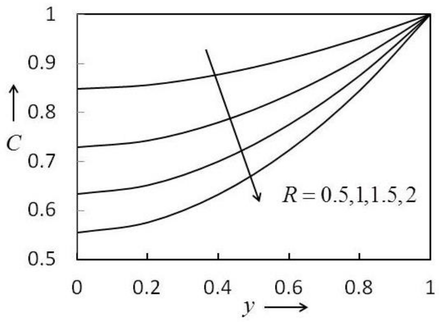

Figure 9: Concentration profile for different R.

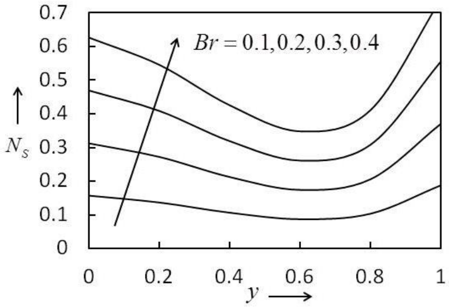

Figure 10: Entropy generation number for different Br.

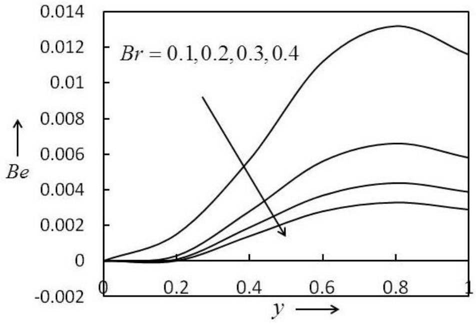

Figure 11: Bejan number for different Br.

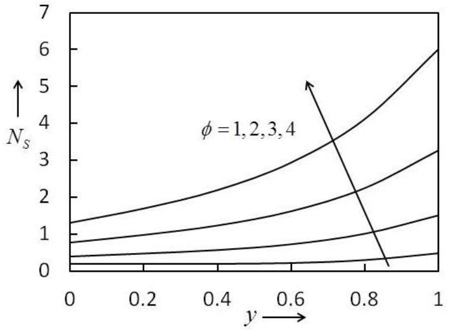

Figure 12: Entropy generation number for different ϕ.

Figures 12 and 13 show that both N_s and Be increases due to increase in ϕ. Figures 14 and 15 show that both N_s and Be decrease when δ increase.

V. CONCLUSIONS

The main conclusions are

1. An increase in drag parameter causes an increase in solid displacement of the two-phase medium, while fluid velocity reduces.
2. An increase in volume fraction parameter of the fluid causes a reduction in solid displacement, whereas fluid velocity accelerated.

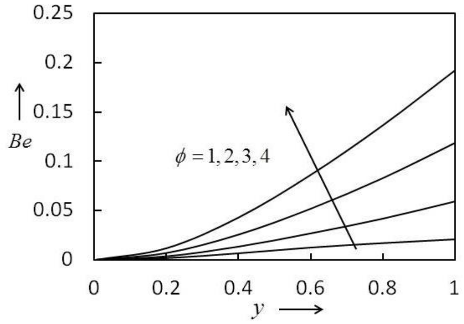

Figure 13: Bejan number for different ϕ.

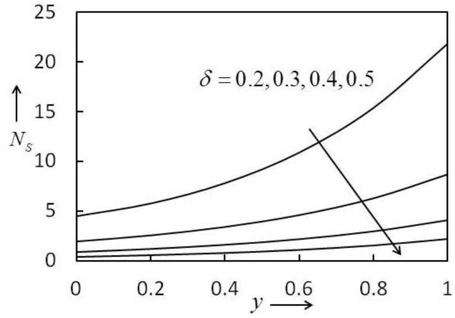

Figure 14: Entropy generation number for different δ.

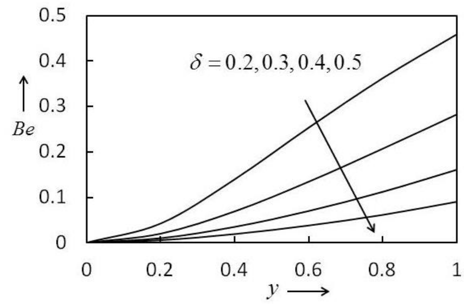

Figure 15: Bejan number for different Pr.

3. Solid displacement and fluid velocity reduces due to increase in pressure gradient.
4. To increase temperature of the system Brinkman number may be increased.
5. To decrease concentration profile of the system chemical reaction parameter may be increased.
6. An increase in Brinkman number and volume fraction parameter increases the rate of entropy generation, while drag parameter decreases the rate.
7. An increase in Brinkman number and drag parameter reduces the value of Bejan number, while it increases with increase in volume fraction parameter.

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NOMENCLATURE

a, width of the layer; B_0, strength of magnetic field; Be, Bejan number; Br, Brinkman number; $\bar{C}$, concentration; $\bar{C}_0$, ambient concentration; C_w, concentration at $\bar{y} = a$; D, molecular diffusivity; Gc, solutal Grashof number;

G_r, thermal Grashof number; K, drag coefficient; K_0, thermal conductivity; M, Hartman number; N_s entropy generation number; $\bar{P}$, pressure; Pe, Peclet number; R, chemical reaction parameter; $\bar{R}$, rate of chemical reaction; Re, Reynolds number; Sc, Schmidt number; $\bar{T}$, temperature; $\bar{T}_0$, ambient temperature; T_w, wall temperature at $\bar{y} = a$; u, non-dimensional solid displacement; $\bar{u}$, solid displacement; V, average velocity; v, non-dimensional fluid velocity; $\bar{v}$, fluid velocity; x, y, non-dimensional Cartesian coordinates; $\bar{x}, \bar{y}$, Cartesian coordinates; β_C, coefficient of mass transfer; β_T, coefficient of heat transfer; δ, viscous drag; η, ratio of the bulk fluid viscosity to the apparent fluid viscosity in porous layer; μ, Lame constant; ϕ, volume fraction component for fluid phase; ρ^f, fluid density; μ_a, apparent viscosity of fluid; μ_f, intrinsic viscosity of fluid; σ, electrical conductivity; Ω, non-dimensional temperature.

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APPENDIX

$$A_1 = \frac{m_2}{m_2 \exp(-m_1) - m_1 \exp(-m_2)},\ A_2 = \frac{Gr - \phi P}{(\delta + M)},$$

$$A_3 = -\frac{GcA_1}{m_1^2 - \mathrm{Re}\,\eta\, m_1 - \eta(\delta + M)},$$

$$A_4 = -\frac{GcA_1}{m_2\left(m_2^2 - \mathrm{Re}\,\eta\, m_2 - \eta(\delta + M)\right)},$$

$$A_{5a} = \frac{1}{m_4 \exp(-m_3) - m_3 \exp(-m_4)},$$

$$A_{5b} = \exp(-m_4)\,(m_1 A_3 + m_2 A_4) - m_4 A_2 - m_4 A_3 \exp(-m_1) - m_4 A_4 \exp(-m_2),$$

$$A_5 = A_{5a} A_{5b},\ A_6 = \frac{m_1 A_3 + m_2 A_4 + m_3 A_5}{m_4},$$

$$A_7 = -\delta\left(\frac{A_6}{m_4} + \frac{A_5}{m_3} + \frac{A_4}{m_2} + \frac{A_3}{m_1}\right),$$

$$A_8 = -\frac{P}{2}(1 - \phi) + \frac{A_5 \delta}{m_3^2}\exp(-m_3) + \frac{A_6 \delta}{m_4^2}\exp(-m_4) + \frac{A_3 \delta}{m_1^2}\exp(-m_1) + \frac{A_4 \delta}{m_2^2}\exp(-m_2) + \frac{A_2 \delta}{2} - A_7,$$

$$A_9=\frac{A_3\delta}{m_1^2},\ A_{10}=\frac{A_4\delta}{m_2^2},\ A_{11}=\frac{A_5\delta}{m_3^2},\ A_{12}=\frac{A_6\delta}{m_4^2},$$
$$A_{13}=\frac{P}{2}(1-\phi)-\frac{A_2\delta}{2},\ A_{14}=-\left(\frac{A_5}{m_3}\right)^2-A_5^2,$$
$$A_{15}=-\left(\frac{A_6}{m_4}\right)^2-A_6^2,\ A_{16}=-\left(\frac{A_3}{m_1}\right)^2-A_3^2,$$
$$A_{17}=-\left(\frac{A_4}{m_2}\right)^2-A_4^2,\ A_{18}=-\frac{2A_5A_6}{m_3m_4}-2A_5A_6,$$
$$A_{19}=-\frac{2A_5A_3}{m_3m_1}-2A_5A_3,\ A_{20}=-\frac{2A_5A_4}{m_3m_2}-2A_5A_4,$$
$$A_{21}=-2A_5A_2,\ A_{22}=-\frac{2A_3A_6}{m_1m_4}-2A_3A_6,$$
$$A_{23}=-\frac{2A_4A_6}{m_2m_4}-2A_4A_6,\ A_{24}=-2A_2A_6,$$
$$A_{25}=-\frac{2A_3A_4}{m_1m_2}-2A_3A_4,\ A_{26}=-2A_2A_3,$$
$$A_{27}=-2A_2A_4,\ A_{28}=\frac{A_{14}}{4m_3^2-2m_3Pe},\ A_{29}=\frac{A_{15}}{4m_4^2-2m_4Pe},$$
$$A_{30}=\frac{A_{16}}{4m_1^2-2m_1Pe},\ A_{31}=\frac{A_{17}}{4m_2^2-2m_2Pe},$$
$$A_{32}=\frac{A_{18}}{(m_3+m_4)^2-(m_3+m_4)Pe},\ A_{33}=\frac{A_{19}}{(m_3+m_1)^2-(m_3+m_1)Pe},$$
$$A_{34}=\frac{A_{20}}{(m_3+m_2)^2-(m_3+m_2)Pe},\ A_{35}=\frac{A_{22}}{(m_4+m_1)^2-(m_4+m_1)Pe},$$
$$A_{36}=\frac{A_{23}}{(m_4+m_2)^2-(m_4+m_2)Pe},\ A_{37}=\frac{A_{25}}{(m_2+m_1)^2-(m_2+m_1)Pe},$$
$$A_{38}=\frac{A_{26}}{m_1^2-m_1Pe},\ A_{39}=\frac{A_{27}}{m_2^2-m_2Pe},\ A_{40}=\frac{A_{21}}{m_3^2-m_3Pe},$$
$$A_{41}=\frac{A_{24}}{m_4^2-m_4Pe},\ A_{42}=\frac{1}{Pe}\Big\{-2m_3A_{28}-2m_4A_{29}-2m_1A_{30}-2m_2A_{31}-(m_3+m_4)A_{32}-(m_3+m_1)A_{33}-(m_3+m_2)A_{34}-(m_1+m_4)A_{35}-(m_2+m_4)A_{36}-(m_1+m_2)A_{37}-m_1A_{38}-m_2A_{39}-m_3A_{40}-m_4A_{41}-\frac{A_2^2}{Pe}\Big\},$$
$$A_{43}=-A_{42}\exp(-Pe)-A_{28}\exp(-2m_3)-A_{29}\exp(-2m_4)-A_{30}\exp(-2m_1)-A_{31}\exp(-2m_2)-A_{32}\exp(-m_3-m_4)-A_{33}\exp(-m_3-m_1)-A_{34}\exp(-m_3-m_2)-A_{35}\exp(-m_4-m_1)-A_{36}\exp(-m_2-m_4)-A_{37}\exp(-m_2-m_1)-A_{38}\exp(-m_1)-A_{39}\exp(-m_2)-A_{40}\exp(-m_3)-A_{41}\exp(-m_4)-\frac{A_2^2}{Pe},$$
$$A_{44}=-\frac{GrA_{42}}{(\delta+M)}+\frac{A_2^2\mathrm{Re}Gr}{Pe(\delta+M)^2},\ A_{45}=\frac{-Gr\ \eta\ A_{43}}{Pe^2-\eta\ \mathrm{Re}\ Pe-(\delta+M)\eta},$$
$$A_{46}=\frac{-Gr\ \eta\ A_{30}}{4m_1^2-2\eta\ \mathrm{Re}\ m_1-(\delta+M)\eta},\ A_{47}=\frac{-Gr\ \eta\ A_{31}}{4m_2^2-2\eta\ \mathrm{Re}\ m_2-(\delta+M)\eta},$$
$$A_{48}=\frac{-Gr\ \eta\ A_{28}}{4m_3^2-2\eta\ \mathrm{Re}\ m_3-(\delta+M)\eta},\ A_{49}=\frac{-Gr\ \eta\ A_{29}}{4m_4^2-2\eta\ \mathrm{Re}\ m_4-(\delta+M)\eta},$$
$$A_{50}=\frac{-Gr\ \eta\ A_{32}}{(m_3+m_4)^2-\eta\ \mathrm{Re}\ (m_3+m_4)-(\delta+M)\eta},$$
$$A_{51}=\frac{-Gr\ \eta\ A_{33}}{(m_3+m_1)^2-\eta\ \mathrm{Re}\ (m_3+m_1)-(\delta+M)\eta},$$
$$A_{52}=\frac{-Gr\ \eta\ A_{34}}{(m_3+m_2)^2-\eta\ \mathrm{Re}\ (m_3+m_2)-(\delta+M)\eta},$$
$$A_{53}=\frac{-Gr\ \eta\ A_{35}}{(m_4+m_1)^2-\eta\ \mathrm{Re}\ (m_4+m_1)-(\delta+M)\eta},$$
$$A_{54}=\frac{-Gr\ \eta\ A_{36}}{(m_4+m_2)^2-\eta\ \mathrm{Re}\ (m_4+m_2)-(\delta+M)\eta},$$
$$A_{55}=\frac{-Gr\ \eta\ A_{37}}{(m_1+m_2)^2-\eta\ \mathrm{Re}\ (m_1+m_2)-(\delta+M)\eta},$$
$$A_{56}=\frac{-Gr\ \eta\ A_{38}}{m_1^2-2\eta\ \mathrm{Re}\ m_1-(\delta+M)\eta},\ A_{57}=\frac{-Gr\ \eta\ A_{39}}{m_2^2-\eta\ \mathrm{Re}\ m_2-(\delta+M)\eta},$$
$$S_1=\frac{-GrA_{42}}{\delta+M}-A_{45}\exp(-Pe)-A_{46}\exp(-2m_1)-A_{47}\exp(-2m_2)-A_{48}\exp(-2m_3)-A_{49}\exp(-2m_4)-A_{50}\exp(-m_3-m_4)-A_{51}\exp(-m_3-m_1)-A_{52}\exp(-m_3-m_2)-A_{53}\exp(-m_1-m_4)-A_{54}\exp(-m_2-m_4)-A_{55}\exp(-m_1-m_2)-A_{56}\exp(m_1)-A_{57}\exp(m_2)+\frac{A_{40}Gr\ \eta\ \exp(m_3)}{m_3^2-\mathrm{Re}\ \eta m_3-(\delta+M)\eta}+\frac{A_{41}Gr\ \eta\ \exp(m_4)}{m_4^2-\mathrm{Re}\ \eta m_4-(\delta+M)\eta}+\frac{A_2^2Gr}{\delta+M},$$
$$S_2=\frac{1}{m_3-m_4},\ S_3=S_1m_3+PA_{45}+2A_{46}m_1+2A_{47}m_2+2A_{48}m_3+2A_{49}m_4+A_{50}(m_3+m_4)+A_{51}(m_3+m_1)+A_{52}(m_3+m_2)+A_{53}(m_1+m_4)-A_{54}(m_2+m_4)+A_{55}(m_1+m_2)+A_{56}m_1+A_{57}m_2-\frac{A_{40}Gr\ \eta\ m_3}{m_3^2-\mathrm{Re}\ \eta m_3-(\delta+M)\eta}-\frac{A_{41}Gr\ \eta\ m_4}{m_4^2-\mathrm{Re}\ \eta m_4-(\delta+M)\eta},$$
$$A_{58}=S_1-S_2S_3-\frac{A_{40}Gr\ \eta}{m_3^2-\mathrm{Re}\ \eta m_3-(\delta+M)\eta},$$
$$A_{59}=S_2S_3-\frac{A_{41}Gr\ \eta}{m_4^2-\mathrm{Re}\ \eta m_4-(\delta+M)\eta},$$
$$A_{60}=-\frac{A_2^2Gr}{Pe(\delta+M)},\ A_{61}=-\frac{\mathrm{Re}A_2^2Gr}{Pe(\delta+M)^2},$$
$$A_{62}=-\delta A_{44}-\delta A_{61},\ A_{63}=\frac{-\delta A_{45}}{Pe},\ A_{64}=\frac{-\delta A_{48}}{m_3},$$
$$A_{65}=\frac{-\delta A_{49}}{2m_4},\ A_{66}=\frac{-\delta A_{46}}{2m_1},\ A_{67}=\frac{-\delta A_{47}}{2m_2},\ A_{68}=\frac{-\delta A_{50}}{m_3+m_4},$$
$$A_{69}=\frac{-\delta A_{51}}{m_3+m_1},\ A_{70}=\frac{-\delta A_{52}}{m_3+m_2},\ A_{71}=\frac{-\delta A_{53}}{m_1+m_4},$$
$$A_{71a}=\frac{-\delta A_{54}}{m_2+m_4},\ A_{72}=\frac{-\delta A_{55}}{m_1+m_2},\ A_{73}=\frac{\delta A_{56}}{-m_1},\ A_{74}=\frac{\delta A_{57}}{-m_2},$$
$$A_{75}=\frac{\delta A_{58}}{-m_3},\ A_{75a}=\frac{\delta A_{59}}{-m_4},\ A_{76}=-\frac{\delta A_{60}}{2},$$
$$A_{77}=-A_{63}-A_{64}-A_{65}-A_{66}-A_{67}-A_{68}-A_{69}-A_{70}-A_{71}-A_{71a}-A_{72}-A_{73}-A_{74}-A_{75}-A_{75a},$$
$$A_{78}=\frac{A_{62}}{2},\ A_{79}=\frac{A_{76}}{2m_3},\ A_{80}=\frac{-A_{63}}{Pe},\ A_{81}=\frac{-A_{64}}{2m_3},$$
$$A_{82}=\frac{-A_{65}}{2m_3},\ A_{83}=\frac{-A_{66}}{2m_1},\ A_{84}=\frac{-A_{67}}{2m_2},\ A_{85}=\frac{-A_{68}}{m_1+m_2},$$
$$A_{86}=\frac{-A_{69}}{m_3+m_1},\ A_{87}=\frac{-A_{70}}{m_3+m_2},\ A_{88}=\frac{-A_{71}}{m_4+m_1},$$
$$A_{89}=\frac{-A_{71a}}{m_4+m_2},\ A_{90}=\frac{-A_{72}}{m_3+m_4},\ A_{91}=\frac{-A_{73}}{m_1},\ A_{92}=\frac{-A_{74}}{m_3},$$
$$A_{93}=\frac{-A_{75}}{m_3},\ A_{94}=\frac{-A_{75a}}{m_4},\ A_{95}=-A_{77}-A_{78}-A_{79}-A_{80}\exp(-Pe)-A_{81}\exp(-2m_1)-A_{82}\exp(-2m_2)-A_{83}\exp(-2m_3)-A_{84}\exp(-2m_4)-A_{85}\exp(-m_1-m_2)-A_{86}\exp(-m_3-m_1)-A_{87}\exp(-m_4-m_1)-A_{88}\exp(-m_2-m_3)-A_{89}\exp(-m_4-m_2)-A_{90}\exp(-m_3-m_4)-A_{91}\exp(-m_1)-A_{92}\exp(-m_2)-A_{93}\exp(-m_3)-A_{94}\exp(-m_4).$$

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